

METRIC CURRENTS AND ALBERTI REPRESENTATIONS

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ABSTRACT. We relate Ambrosio-Kirchheim metric currents to Alberti representations and Weaver derivations. In particular, given a metric current T , we show that if the module $\mathcal{X}(\|T\|)$ of Weaver derivations is finitely generated, then T can be represented in terms of derivations; this extends previous results of Williams. Applications of this theory include an approximation of 1-dimensional metric currents in terms of normal currents and the construction of Alberti representations in the directions of vector fields.

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1. INTRODUCTION

Overview. The goal of this paper is to relate metric currents to Alberti representations and Weaver derivations. In particular, it seems that metric currents carry a weak notion of a differentiable structure which we try to describe by using Alberti representations and Weaver derivations. As a first application we prove an approximation result in which a 1-dimensional metric current is approximated by a sequence of normal currents. As a second application we show how to use 1-dimensional normal currents to produce Alberti representations in the directions of vector fields.

Metric currents. Federer and Fleming [FF60] introduced the theory of currents to study the Plateau problem in Euclidean spaces of dimension higher than 2, and overtime currents have proven useful to attack a wide range of problems, see [ABL88, Lin99, GMS89] to cite some examples. In order to study similar problems in general metric spaces, it became desirable to have an analogue of the Federer-Fleming currents and a major obstacle was that the classical definition of currents

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uses the differentiable structure of \mathbb{R}^N . In [AK00] Ambrosio and Kirchheim, inspired by an idea of de Giorgi [DG95], developed a theory of metric currents starting by circumventing the lack of a differentiable structure. Essentially, k -dimensional metric currents are defined by duality with $(k+1)$ -tuples of Lipschitz functions (f, π_1, \dots, π_k) , where the first function f is also bounded. The axioms that currents satisfy are then designed so that one can formally treat, to some extent, the $(k+1)$ -tuple (f, π_1, \dots, π_k) as a k -dimensional differential form $fd\pi_1 \wedge \dots \wedge d\pi_k$. In [Wil10] Williams showed that in a differentiability space (X, μ) , those metric currents whose masses are absolutely continuous with respect to μ are dual to the differential k -forms defined using the differentiable structure. This result was the starting point of the present work in which, roughly speaking, we remove the assumption that (X, μ) is a differentiability space.

For a treatment of metric currents we refer the reader to [AK00]; some basic facts are recalled in Subsection 2.1. Note that Lang [Lan11] has formulated an alternative theory of metric currents in which the finite mass axiom is removed; our results have natural counterparts in that setting.

Alberti representations. Alberti representations were introduced in [Alb93] to prove the rank-one property for BV functions; they were later applied to study the differentiability properties of Lipschitz functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$ [ACP05, ACP10] and have recently been used to obtain a description of measures in differentiability spaces [Bat12]. We give here an informal definition and refer the reader to [Bat12, Sch13] and Subsection 2.2 for further details.

An **Alberti representation** of a Radon measure μ is a generalized Lebesgue decomposition of μ in terms of rectifiable measures supported on path fragments; a **path fragment** in X is a Lipschitz map $\gamma : K \rightarrow X$ where $K \subset \mathbb{R}$ is compact; the set of fragments in X will be denoted by $\text{Frag}(X)$ and topologized as a subspace of $K(X)$, the set of compact subsets of X with the Vietoris topology. An Alberti representation of μ is then a decomposition:

$$(1.1) \quad \mu = \int_{\text{Frag}(X)} \nu_\gamma dP(\gamma),$$

where P is a regular Borel probability measure on $\text{Frag}(X)$, and ν associates to each fragment γ a finite Radon measure ν_γ which is absolutely continuous with respect to the 1-dimensional Hausdorff measure \mathcal{H}^1_γ on the image of γ . Examples of an Alberti representation are offered by Fubini's Theorem; however, in general it is necessary to work with path fragments instead of Lipschitz curves because the space X on which μ is defined might lack any rectifiable curve.

Weaver derivations and their relationship with Alberti representations.

Weaver derivations, hereafter simply called derivations, were introduced in [Wea00] and provide a quite broad framework to formulate a notion of differentiability on metric measure spaces. To fix the ideas, let $\text{Lip}(X)$ denote the set of real-valued Lipschitz functions defined on X and let $\text{Lip}_b(X) \subset \text{Lip}(X)$ denote the subset of bounded Lipschitz functions. The vector space $\text{Lip}_b(X)$ becomes a Banach algebra with norm:

$$(1.2) \quad \|f\|_{\text{Lip}_b(X)} = \max(\|f\|_\infty, \mathbf{L}(f)),$$

where $\mathbf{L}(f)$ denotes the Lipschitz constant of f . It is a fact [Wea99, Ch. 2] that the Banach algebra $\text{Lip}_b(X)$ is a dual Banach space and so it has a weak* topology;

for the present work, it is sufficient to consider sequential convergence which is characterized as follows: $f_n \xrightarrow{w^*} f$ if and only if the global Lipschitz constants of the f_n are uniformly bounded and $f_n \rightarrow f$ pointwise.

Having fixed a Radon measure μ on X , derivations are weak* continuous bounded linear maps $D : \text{Lip}_b(X) \rightarrow L^\infty(\mu)$ which satisfy the product rule $D(fg) = fDg + gDf$. Intuitively, derivations can be interpreted as *measurable vector fields* and depend only on the measure class of μ . For example, if \mathcal{L}^n denotes the Lebesgue measure on \mathbb{R}^n , one obtains a derivation $\frac{\partial}{\partial x_i} : \text{Lip}_b(X) \rightarrow L^\infty(\mathcal{L}^n)$ by taking the partial derivative of Lipschitz functions in the x_i -direction. Note that the set of derivations is an $L^\infty(\mu)$ -module.

Even for metric measure spaces (X, μ) which cannot admit a differentiable structure the module $\mathcal{X}(\mu)$ can be nontrivial. Moreover, one can also study the modules $\mathcal{X}(\mu)$ and $\mathcal{X}(\mu')$ for mutually singular measures μ and μ' on the *same space* X . Derivations provide thus a broad definition of differentiability for Lipschitz functions and it is desirable to obtain a characterization of derivations for general metric measure spaces. In [Sch13] the author showed that there is a correspondence between Alberti representations and Weaver derivations which implies, roughly speaking, that derivations are obtained by taking derivatives along fragments. Some results in [Sch13] relevant for the present work are recalled in Subsection 2.4.

Main results. We now describe the main results of this paper and refer the reader to the following sections for an explanation of the terminology; we denote by $\mathbf{M}_k(X)$ the Banach space of k -dimensional metric currents in the metric space X .

It is an observation¹ that there is a close similarity between Weaver derivations and 1-dimensional metric currents (see Sec. 3). In the light of [Sch13] it is thus natural to ask how this similarity relates to the existence of Alberti representations. We show that the mass $\|T\|$ of a k -dimensional metric current T possesses Alberti representations in the directions of k -dimensional cone fields. Specifically, in Section 4 we prove the following:

Theorem 1.3. *Let $T \in \mathbf{M}_k(X) \setminus \{0\}$ for $k > 0$. Then there are disjoint Borel sets $\{V_j\}_j$ and 1-Lipschitz functions $\pi^j : X \rightarrow \mathbb{R}^k$ (on \mathbb{R}^k we consider the l^∞ norm) such that:*

- (1) $\|T\| \left(X \setminus \bigcup_j V_j \right) = 0$.
- (2) *For all $\varepsilon > 0$ and for any k -dimensional cone field \mathcal{C} , the measure $\|T\|$ admits a $(1, 1 + \varepsilon)$ -biLipschitz Alberti representation \mathcal{A} with $\mathcal{A} \ll V_j$ in the π^j -direction of \mathcal{C} .*

In particular, the module $\mathcal{X}(\|T\|)$ contains k independent derivations.

Note that the proof of Theorem 1.3 actually does not take full advantage of the *joint continuity* of T in its last arguments (π_1, \dots, π_k) and so applies to a larger class of metric functionals. It might be worth mentioning a connection between Theorem 1.3 and the classical Rademacher Theorem, which asserts that a Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at \mathcal{H}^n -a.e. point, where \mathcal{H}^n denotes the Lebesgue measure. Given a top dimensional current $T \in \mathbf{M}_n(\mathbb{R}^n)$, Theorem 1.3 implies that the mass measure $\|T\|$ possesses n -independent Alberti representations, and then it follows that the conclusion of Rademacher's Theorem holds for the measure $\|T\|$. Recently, M. Csörnyei and P. Jones have announced that Rademacher's

¹Gong [Gon11, pg. 3] attributes it to Wenger

Theorem is *sharp* in the sense that, if its *conclusion* holds for the metric measure space (\mathbb{R}^n, μ) , then μ must be absolutely continuous with respect to the Lebesgue measure.

Note also that Theorem 1.3 suggests that metric currents come with some weak notion of a *differentiable structure*. To make this intuition precise, we prove a representation formula for metric currents in terms of Weaver derivations; essentially, a k -dimensional metric current T is of the form $\omega_T \|T\|$, where ω_T is a measurable k -dimensional vector field (see the next Subsection) and the formal k -form (f, π_1, \dots, π_k) can be interpreted as a k -form in the k -th exterior power of the Weaver's cotangent bundle (see also the next Subsection). Specifically, in Section 5 we prove:

Theorem 1.4. *Let $T \in \mathbf{M}_k(X)$ and assume that $\mathcal{X}(\|T\|)$ is finitely generated with N generators. Then there is $\omega_T \in \mathcal{X}^k(\|T\|)$ (or $\omega_T \in \text{Ext}_{\|T\|}^k \mathcal{X}(\|T\|)$ or $\omega_T \in \text{Ext}^k \mathcal{X}(\|T\|)$) such that:*

$$(1.5) \quad T(f, \pi_1, \dots, \pi_k) = \int_X f \langle \omega_T, d\pi_1 \wedge \dots \wedge d\pi_k \rangle d\|T\|.$$

Moreover, ω_T has norm at most $(C(N))^k \binom{N}{k}$.

Note that the assumption that $\mathcal{X}(\mu)$ is finitely generated is not very restrictive as it holds if $\|T\|$ is doubling or if the ambient metric space is doubling [Sch13]. Note also how Theorem 1.4 parallels the representation of classical currents ([KP08, Sec. 7.2], [Fed69, Sec. 4.1]).

In Section 6 we provide two applications of this theory. The first application provides an approximation of 1-dimensional metric currents in terms of normal currents:

Theorem 1.6. *If $T \in \mathbf{M}_1(Z)$ where Z is a Banach space and if the module $\mathcal{X}(\|T\|)$ is finitely generated, then there is a sequence of normal currents $\{N_n\} \subset \mathbf{N}_1(Z)$ such that:*

$$(1.7) \quad \lim_{n \rightarrow \infty} \|T - N_n\|_{\mathbf{M}_1(Z)} = 0.$$

This provides an affirmative answer to the 1-dimensional case of a question raised in [AK00, pg. 68]. Note that even though we prove the result in Banach spaces, the proof can be adapted to spaces where fragments can be *filled-in* to give Lipschitz curves. In particular, the structure of 1-dimensional metric currents seems very close to that of normal currents. Note that this is not the case for classical currents.

As a second application we provide a different method to produce Alberti representations which is based on results of Paolini and Stepanov [PS12, PS13] on the structure of 1-dimensional normal currents. This approach allows to gain a better control on the direction of the Alberti representations; in fact, instead of obtaining Alberti representations in the ψ -direction of a finite dimensional cone field \mathcal{C} , one obtains Alberti representations in the ψ -direction of a *vector field* v . Moreover, the Lipschitz function ψ can be taken to be l^2 -valued, allowing to control countably many functions. The precise result is Theorem 6.31, which is proved in Subsection 6.2. This result is based on identifying a special class of derivations, which we call **normal derivations**, which have properties closely related to those of normal currents. A further direction related to this result is to extend the action of derivations to Lipschitz functions which take values in Banach spaces with the Radon-Nikodym property: this will be pursued elsewhere.

Technical tools. Section 7 contains some technical results. In Subsection 7.1 we discuss exterior powers in the categories of Banach spaces, $L^\infty(\mu)$ -modules and $L^\infty(\mu)$ -normed modules. This material is just an adaptation of the treatment in [CLM79, Ch. 2 and 3] of tensor products. The motivation is to give a precise meaning to an exterior product of derivations $D_1 \wedge \cdots \wedge D_k$; as $\mathcal{X}(\mu)$ is an $L^\infty(\mu)$ -normed module, the construction can be done in the three aforementioned categories and the results are different. In the author's opinion, the most natural choice is probably that of $L^\infty(\mu)$ -normed modules.

In Subsection 7.2 we prove Theorem 7.97 which is a criterion to produce Alberti representations for measures in Banach spaces when the direction and the speed are specified by linear maps. This result is used in the proof of Theorem 1.6.

In Subsection 7.3 we discuss Theorem 7.101 which is a renorming trick which allows to obtain a strictly convex local norm on $\mathcal{X}(\mu)$ by taking a biLipschitz deformation of the metric on the ambient metric space. This result is used in the proof of Theorem 6.31 and might be of independent interest. It is worth to point out that Theorem 7.101, when specialized to the context of differentiability spaces, gives a stronger conclusion than Cheeger's renorming Theorem [Che99, Sec. 12] for PI-spaces. In fact, Theorem 7.101 works in general differentiability spaces, does require only a small perturbation of the distance function, and works globally (while Cheeger's argument works only on a single chart).

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2. PRELIMINARIES

2.1. Metric currents. We recall here some definitions and facts about metric currents and refer the reader to [AK00, Lan11] for more information.

Let $\mathcal{D}^k(X)$ denote the set of $\text{Lip}_b(X) \times (\text{Lip}(X))^{k-1}$ of $(k+1)$ -tuples of Lipschitz functions where the first one is bounded. Intuitively, we want to think of a $(k+1)$ -tuple (f, π_1, \dots, π_k) as a k -differential form $f d\pi_1 \wedge \cdots \wedge d\pi_k$. A map $T : V \rightarrow \mathbb{R}$, where V is a vector space over \mathbb{R} is called **subadditive** if for each $v_1, v_2 \in V$ one has:

$$(2.1) \quad |T(v_1 + v_2)| \leq |T(v_1)| + |T(v_2)|;$$

the map T is called **positively 1-homogeneous** if for all $(v, \lambda) \in V \times [0, \infty)$ one has:

$$(2.2) \quad |T(\lambda v)| = \lambda |T(v)|.$$

Definition 2.3. A k -dimensional metric functional T on the metric space X is a map $T : \mathcal{D}^k(X) \rightarrow \mathbb{R}$ which is subadditive and positively 1-homogeneous in each of its arguments (f, π_1, \dots, π_k) . The **boundary** ∂T of a k -dimensional metric functional ($k \geq 1$) is the $(k-1)$ -dimensional metric functional defined by:

$$(2.4) \quad \partial T(f, \pi_1, \dots, \pi_{k-1}) = T(1, f, \pi_1, \dots, \pi_{k-1}).$$

For 0-dimensional metric functionals we convene that the boundary is 0.

²for $k = 0$ we let $\mathcal{D}^0(X) = \text{Lip}_b(X)$

Definition 2.5. A k -dimensional metric functional T has finite mass if there is a finite Radon measure μ such that for each $(f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(X)$:

$$(2.6) \quad |T(f, \pi_1, \dots, \pi_k)| \leq \prod_{i=1}^k \mathbf{L}(\pi_i) \int_X |f| d\mu.$$

In this case there is a *minimal* μ satisfying (2.6), called the **mass** of T and denoted by $\|T\|$.

Remark 2.7. Note that any metric functional T with finite mass can be uniquely extended to a map $T : \mathcal{B}^\infty(X) \times (\text{Lip}(X))^k$ so that the first argument f can be taken to be a bounded Borel function.

Definition 2.8. Let T be a k -dimensional metric functional with finite mass. Suppose that $l \leq k$ and that

$$(2.9) \quad \omega = (\psi, \pi_1, \dots, \pi_l) \in \mathcal{B}^\infty(X) \times (\text{Lip}(X))^l;$$

the **restriction** $T \llcorner \omega$ is the $(k-l)$ -dimensional metric functional defined by:

$$(2.10) \quad T \llcorner \omega(f, \tilde{\pi}_1, \dots, \tilde{\pi}_{k-l}) = T(f\psi, \pi_1, \dots, \pi_l, \tilde{\pi}_1, \dots, \tilde{\pi}_{k-l}).$$

In the Introduction we recalled the notion of weak* convergence for sequences in $\text{Lip}_b(X)$. We now introduce a notion of convergence for sequences in $\text{Lip}(X)$ which plays a fundamental rôle in the definition of metric currents: if $\{f_n\} \subset \text{Lip}(X)$ and $f \in \text{Lip}(X)$, we write $f_n \xrightarrow{w^*} f$ if $f_n \rightarrow f$ pointwise and $\sup_n \mathbf{L}(f_n) < \infty$.

Definition 2.11. A k -dimensional metric functional T of finite mass is called a **metric current** if it satisfies the following additional properties³:

- (1) T is multilinear in its arguments f, π_1, \dots, π_k ;
- (2) T is alternating in its last k -arguments: π_1, \dots, π_k ;
- (3) T is **local** in the sense that if some π_i is constant on the set $\{x : f(x) \neq 0\}$, then

$$(2.12) \quad T(f, \pi_1, \dots, \pi_k) = 0;$$

- (4) if $f_n \xrightarrow{w^*} f$ and for $i \in \{1, \dots, k\}$ $\pi_{i,n} \xrightarrow{w^*} \pi_i$, one has:

$$(2.13) \quad \lim_{n \rightarrow \infty} T(f_n, \pi_{i,1}, \dots, \pi_{i,k}) = T(f, \pi_1, \dots, \pi_k).$$

The set of k -dimensional metric currents is denoted by $\mathbf{M}_k(X)$ and is a Banach space with norm $\|T\|_{\mathbf{M}_k(X)} = \|T\|(X)$. An important class of metric currents consists of the normal currents:

Definition 2.14. A k -dimensional metric current is a **normal current** if the boundary ∂T is a metric current. The set of k -dimensional normal currents is denoted by $\mathbf{N}_k(X)$ and is a Banach space with norm:

$$(2.15) \quad \|T\|_{\mathbf{N}_k(X)} = \|T\|(X) + \|\partial T\|(X).$$

³in this formulation some axioms are redundant, see [AK00, Sec. 3].

2.2. Alberti representations. In this Subsection we recall some facts about Alberti representations.

Definition 2.16. Let μ be a Radon measure on a metric space X and $M(X)$ denote the set of finite Radon measures on X ; an Alberti representation of μ is a pair (P, ν) :

- (1) The measure P is a regular Borel probability measure on $\text{Frag}(X)$;
- (2) The map $\nu : \text{Frag}(X) \rightarrow M(X)$ is Borel ⁴ and $\nu_\gamma \ll \mathcal{H}^1_\gamma$;
- (3) The measure μ can be represented as $\mu = \int_{\text{Frag}(X)} \nu_\gamma dP(\gamma)$;
- (4) For each Borel set $A \subset X$ and for all real numbers $a \geq b$, the map $\gamma \mapsto \nu_\gamma(A \cap \gamma(\text{dom } \gamma \cap [a, b]))$ is Borel.

Remark 2.17. Note that in this paper the definition of fragments is different from that used in [Sch13] because, for a fragment $\gamma : K \rightarrow X$, we neither require γ to be biLipschitz nor we require K to have positive Lebesgue measure. However, an application of the area formula [Kir94, Cor. 8] shows that the results that we cite from [Sch13] are still valid in this setting.

In order to define notions of speed and direction for Alberti representations we recall the definitions of Euclidean cone and of the metric differential of a fragment.

Definition 2.18. Let $\alpha \in (0, \pi/2)$, $w \in \mathbb{S}^{n-1}$; the **open cone** $\mathcal{C}(w, \alpha) \subset \mathbb{R}^n$ with axis w and opening angle α is:

$$(2.19) \quad \mathcal{C}(w, \alpha) = \{u \in \mathbb{R}^q : \tan \alpha \langle w, u \rangle > \|\pi_w^\perp u\|_2\},$$

where π_w^\perp denotes the orthogonal projection on the orthogonal complement of the line $\mathbb{R}w$.

Definition 2.20. For a fragment $\gamma \in \text{Frag}(X)$, the metric differential $\text{md } \gamma(t)$ of γ at $t \in \text{dom } \gamma$ is the limit

$$(2.21) \quad \lim_{\text{dom } \gamma \ni t' \rightarrow t} \frac{d(\gamma(t'), \gamma(t))}{|t' - t|}$$

whenever it exists; if t is an isolated point of $\text{dom } \gamma$ we convene that the limit is 0.

In order to measure the direction of a fragment γ , one uses a Lipschitz function $f : X \rightarrow \mathbb{R}^q$ and studies the direction of $(f \circ \gamma)'$ using cones.

Definition 2.22. An n -dimensional **cone field** \mathcal{C} is a Borel map from X to the set of open cones in \mathbb{R}^n . Alternatively, an n -dimensional cone-field \mathcal{C} is specified by a pair of Borel maps $\alpha : X \rightarrow (0, \pi/2)$ and $w : X \rightarrow \mathbb{S}^{n-1}$ by letting $\mathcal{C}(x) = \mathcal{C}(\alpha(x), w(x))$.

Given a Lipschitz function $f : X \rightarrow \mathbb{R}^n$, an Alberti representation $\mathcal{A} = (P, \nu)$ is said to be **in the f -direction of the n -dimensional cone-field \mathcal{C}** if for P -a.e. $\gamma \in \text{Frag}(X)$ and $\mathcal{L}^1 \llcorner \text{dom } \gamma$ -a.e. t one has $(f \circ \gamma)'(t) \in \mathcal{C}(\gamma(t))$.

Definition 2.23. Let $\sigma : X \rightarrow [0, \infty)$ be Borel and $f : X \rightarrow \mathbb{R}$ be Lipschitz. An Alberti representation $\mathcal{A} = (P, \nu)$ is said to be **have f -speed $\geq \sigma$ (resp. $> \sigma$)** if for P -a.e. $\gamma \in \text{Frag}(X)$ and $\mathcal{L}^1 \llcorner \text{dom } \gamma$ -a.e. t one has $(f \circ \gamma)'(t) \geq \sigma(\gamma(t)) \text{md } \gamma(t)$ (resp. $(f \circ \gamma)'(t) > \sigma(\gamma(t)) \text{md } \gamma(t)$).

One finally needs also to control the Lipschitz constant of the fragments used to produce Alberti representations.

⁴on $M(X)$ one takes the weak* topology

Definition 2.24. An Alberti representation $\mathcal{A} = (P, \nu)$ is said to be **C -Lipschitz** (resp. **(C, D) -biLipschitz**) if P -a.e. γ is C -Lipschitz (resp. (C, D) -biLipschitz).

Alberti representations are produced using Rainwater's Lemma [Rai69], which can be regarded as a generalization of the Radon-Nikodym Theorem. In particular, one studies a notion of *nullity for sets* with respects to a *family of measures*.

Definition 2.25. Let $S \subset X$ and $\Omega \subset \text{Frag}(X)$. The set S is said to be Ω -null if for each $\gamma \in \Omega$ one has $\mathcal{H}^1_\gamma(S) = 0$.

We will use the previous notion of nullity mainly for the following families of fragments:

Definition 2.26. Let $f : X \rightarrow \mathbb{R}^n$ and $g : X \rightarrow \mathbb{R}$ be Lipschitz functions, $\sigma : X \rightarrow [0, \infty)$ a Borel function and \mathcal{C} an n -dimensional cone field. We denote by $\text{Frag}(X, f, \mathcal{C}, g, > \sigma)$ the set of those $\gamma \in \text{Frag}(X)$ satisfying:

$$(2.27) \quad (f \circ \gamma)'(t) \in \mathcal{C}(\gamma(t)) \quad \text{for } \mathcal{L}^1 \llcorner \text{dom } \gamma \text{-a.e } t;$$

$$(2.28) \quad (g \circ \gamma)'(t) > \sigma(\gamma(t)) \text{ md } \gamma(t) \quad \text{for } \mathcal{L}^1 \llcorner \text{dom } \gamma \text{-a.e } t;$$

the set $\text{Frag}(X, f, \mathcal{C}, g, \geq \sigma)$ is defined by changing the strict inequality in (2.28) to a non-strict inequality.

The following Theorem (Theorem 2.64 in [Sch13]) is a standard criterion to produce Alberti representations:

Theorem 2.29. *Let X be a complete separable metric space and μ a Radon measure on X . Then the following are equivalent:*

- (1) *The measure μ admits an Alberti representation in the f -direction of \mathcal{C} with g -speed $> \sigma$;*
- (2) *For each $\varepsilon > 0$ the measure μ admits a $(1, 1 + \varepsilon)$ -biLipschitz Alberti representation in the f -direction of \mathcal{C} with g -speed $> \sigma$;*
- (3) *Any Borel set $S \subset X$ which is $\text{Frag}(X, f, \mathcal{C}, g, > \sigma)$ -null is also μ -null.*

In the following we will also use a gluing principle for Alberti representations (compare Theorem 2.46 in [Sch13]). Note that if $\mathcal{A} = (P, \nu)$ is an Alberti representation of μ , for a Borel $U \subset X$ one has an Alberti representation $\mathcal{A} \llcorner U = (P, \nu \llcorner U)$ which is called the **restriction of \mathcal{A} to U** .

Definition 2.30. A countable collection $\{U_\alpha\}$ of μ -measurable sets with positive μ -measure is called an $L^\infty(\mu)$ -**partition of unity** if $\mu((\bigcup_\alpha U_\alpha)^c) = 0$; note that in this case

$$(2.31) \quad \sum_\alpha \chi_{U_\alpha} = 1$$

where convergence of the series is understood in the weak* sense. If the sets U_α are Borel (resp. compact) the $L^\infty(\mu)$ -partition of unity is called **Borel (resp. compact)**.

Theorem 2.32. *Let X be a complete separable metric space and μ a Radon measure on X and $\{U_\alpha\}$ a Borel $L^\infty(\mu)$ -partition of unity. If for each α the measure $\mu \llcorner U_\alpha$ admits an Alberti representation in the f_α -direction of an N_α -dimensional cone field \mathcal{C}_α with g_α -speed $\geq \sigma_\alpha$, then μ admits an Alberti representation \mathcal{A} such that each restriction $\mathcal{A} \llcorner U_\alpha$ is in the f_α -direction of an N_α -dimensional cone field \mathcal{C}_α with g_α -speed $\geq \sigma_\alpha$. Moreover, for each $\varepsilon > 0$ the Alberti representation \mathcal{A} can be assumed to be $(1, 1 + \varepsilon)$ -biLipschitz.*

2.3. Derivations. An $L^\infty(\mu)$ -**module** M is a Banach space M which is also an $L^\infty(\mu)$ -module and such that for all $(m, \lambda) \in M \times L^\infty(\mu)$ one has:

$$(2.33) \quad \|\lambda m\|_M \leq \|\lambda\|_{L^\infty(\mu)} \|m\|_M.$$

Among $L^\infty(\mu)$ -modules a special rôle is played by $L^\infty(\mu)$ -**normed modules**:

Definition 2.34. An $L^\infty(\mu)$ -module M is said to be an $L^\infty(\mu)$ -**normed module** if there is a map

$$(2.35) \quad |\cdot|_{M,\text{loc}} : M \rightarrow L^\infty(\mu)$$

such that:

- (1) For each $m \in M$ one has $|m|_{M,\text{loc}} \geq 0$;
- (2) For all $c_1, c_2 \in \mathbb{R}$ and $m_1, m_2 \in M$ one has:

$$(2.36) \quad |c_1 m_1 + c_2 m_2|_{M,\text{loc}} \leq |c_1| |m_1|_{M,\text{loc}} + |c_2| |m_2|_{M,\text{loc}};$$

- (3) For each $\lambda \in L^\infty(\mu)$ and each $m \in M$, one has:

$$(2.37) \quad |\lambda m|_{M,\text{loc}} = |\lambda| |m|_{M,\text{loc}};$$

- (4) The local seminorm $|\cdot|_{M,\text{loc}}$ can be used to reconstruct the norm of any $m \in M$:

$$(2.38) \quad \|m\|_M = \| |m|_{M,\text{loc}} \|_{L^\infty(\mu)}.$$

Let μ be a Radon measure on the metric space X and denote by $\mathbf{M}_k(\mu)$ the set of k -dimensional metric currents whose mass is absolutely continuous with respect to μ .

Lemma 2.39. *The set $\mathbf{M}_k(\mu)$ is a Banach space and an $L^\infty(\mu)$ -module. It is not an $L^\infty(\mu)$ -normed module if*

- (1) $k > 0$ and $\mathbf{M}_k(\mu) \neq \{0\}$;
- (2) $k = 0$ and μ is not a Dirac measure.

Proof. The space $\mathbf{M}_k(X)$ is a Banach space with the mass norm. Suppose that

$$(2.40) \quad \lim_{k \rightarrow \infty} \|T_k - T\|(X) = 0,$$

and that for each k one has $\|T_k\|(A) = 0$; then one has $\|T\|(A) = 0$. Thus, $\mathbf{M}_k(\mu)$ is a closed subspace of $\mathbf{M}_k(X)$ and hence a Banach space.

The action of $L^\infty(\mu)$ on $\mathbf{M}_k(\mu)$ is given by

$$(2.41) \quad \lambda.T = T \llcorner \lambda,$$

and $\|T \llcorner \lambda\|(X) \leq \|\lambda\|_{L^\infty(\mu)} \|T\|(X)$; thus $\mathbf{M}_k(\mu)$ is an $L^\infty(\mu)$ -module.

Let δ_x denote the Dirac measure concentrated at x . Using [AK00, (iii) in Thm. 3.5] it follows that $\mathbf{M}_k(\delta_x) = 0$ for $k > 0$. Thus, if $T \in \mathbf{M}_k(\mu)$ is non-trivial, there is a Borel $U \subset X$ with

$$(2.42) \quad \|T\|(U), \|T\|(X \setminus U) > 0;$$

in particular,

$$(2.43) \quad \|T\|(X) > \max(\|T \llcorner \chi_U\|(X), \|T \llcorner (1 - \chi_U)\|(X))$$

and so $\mathbf{M}_k(\mu)$ is not an $L^\infty(\mu)$ -normed module.

The same argument can be applied if $k = 0$ and μ is not a Dirac measure. \square

We now introduce the notion of derivations. In the Introduction we described sequential convergence for the weak* topology on $\text{Lip}_b(X)$; for further information we refer the reader to [Wea99, Ch. 2].

Definition 2.44. A derivation $D : \text{Lip}_b(X) \rightarrow L^\infty(\mu)$ is a weak* continuous, bounded linear map satisfying the product rule:

$$(2.45) \quad D(fg) = fDg + gDf.$$

Note that the product rule implies that $Df = 0$ if f is constant. The collection of all derivations $\mathcal{X}(\mu)$ is an $L^\infty(\mu)$ -normed module [Wea00, Thm. 2] and the corresponding local norm will be denoted by $|\cdot|_{\mathcal{X}(\mu), \text{loc}}$. Note also that $\mathcal{X}(\mu)$ depends only on the measure class of μ .

Remark 2.46. Consider a Borel set $U \subset X$ and a derivation $D \in \mathcal{X}(\mu \llcorner U)$. The derivation D can be also regarded as an element of $\mathcal{X}(\mu)$ by extending Df to be 0 on $X \setminus U$ (compare Lemma 2.47). In particular, the module $\mathcal{X}(\mu \llcorner U)$ can be naturally identified with the submodule $\chi_U \mathcal{X}(\mu)$ of $\mathcal{X}(\mu)$.

Derivations are local in the following sense ([Wea00, Lem. 27]):

Lemma 2.47. *If U is μ -measurable and if $f, g \in \text{Lip}_b(X)$ agree on U , then for each $D \in \mathcal{X}(\mu)$, $\chi_U Df = \chi_U Dg$.*

Note that locality allows to extend the action of derivations on Lipschitz functions so that if $f \in \text{Lip}(X)$ and $D \in \mathcal{X}(\mu)$, Df is well-defined (see Remark 2.115 in [Sch13]). We now pass to consider some algebraic properties of $\mathcal{X}(\mu)$.

In general, even if the module $\mathcal{X}(\mu)$ is finitely generated, it is not free. Nevertheless, it is possible to obtain a decomposition into free modules over *smaller rings* [Wea00, Sch]:

Theorem 2.48. *Suppose that the module $\mathcal{X}(\mu)$ is finitely generated with N generators. Then there is a Borel partition $X = \bigcup_{i=1}^N X_i$ such that, if $\mu(X_i) > 0$, then $\mathcal{X}(\mu \llcorner X_i)$ is free of rank i as an $L^\infty(\mu \llcorner X_i)$ -module. A basis of $\mathcal{X}(\mu \llcorner X_i)$ will be called a **local basis of derivations**.*

In many applications in Analysis on metric spaces the assumption that $\mathcal{X}(\mu)$ is finitely generated is not restrictive: for example it holds if either μ or X are doubling (Corollary 5.136 in [Sch13]).

In practice, to explicitly use the linear independence of some derivations it is useful to construct *pseudodual* Lipschitz functions:

Definition 2.49. We say that Lipschitz functions $\{g_j\}_{j=1}^k \subset \text{Lip}_b(X)$ are **pseudodual to $\{D_i\}_{i=1}^k \subset \mathcal{X}(\mu)$ on a Borel set U** , if $\chi_U(D_i g_j - \delta_{i,j}) = 0$ and $\mu(U) > 0$. In this case, note that the derivations $\{\chi_U D_i\}_{i=1}^k \subset \mathcal{X}(\mu \llcorner U)$ are independent⁵.

The following Lemma constructs pseudodual functions given independent derivations. However, it is a slight improvement of similar results [Gon12, Sch] because it controls the norm of the derivations obtained. This improvement is used in the proof of Theorem 1.4.

Lemma 2.50. *Suppose that the derivations $\{D_i\}_{i=1}^k \subset \mathcal{X}(\mu)$ are independent. Then there are a Borel $L^\infty(\mu)$ -partition of unity V_α and there are, for each α , derivations $\{D_{\alpha,i}\}_{i=1}^k \subset \chi_{V_\alpha} \mathcal{X}(\mu)$ and 1-Lipschitz functions $\{g_{\alpha,j}\}_{j=1}^k \subset \text{Lip}_b(X)$ such that:*

⁵we consider the ring $L^\infty(\mu \llcorner U)$

- (1) The submodule of $\mathcal{X}(\mu)$ generated by the derivations $\{D_{\alpha,i}\}_{i=1}^k$ contains the submodule generated by the derivations $\{\chi_{U_\alpha} D_i\}_{i=1}^k$;
- (2) The derivations $\{D_{\alpha,i}\}_{i=1}^k$ have norm at most $C(k)$, a universal constant depending only on k ;
- (3) The functions $\{g_{\alpha,j}\}_{j=1}^k$ are pseudodual to the derivations $\{D_{\alpha,i}\}_{i=1}^k$ on V_α .

To prove Lemma 2.50 we introduce a notion of normalization for derivations. We first consider the set where a given derivation vanishes:

Definition 2.51. Given a derivation $D \in \mathcal{X}(\mu)$, having chosen a Borel representative of $|D|_{\mathcal{X}(\mu),\text{loc}}$, we let

$$(2.52) \quad N_D = \left\{ x : |D|_{\mathcal{X}(\mu),\text{loc}}(x) = 0 \right\};$$

note that N_D is well-defined up to Borel μ -null sets and that $\lambda D = 0$ iff $\lambda \in \chi_{N_D} L^\infty(\mu)$. If N_D is μ -null, we say that D is **nowhere vanishing**.

Lemma 2.53. For a derivation $D \in \mathcal{X}(\mu)$, having chosen a Borel representative of $|D|_{\mathcal{X}(\mu),\text{loc}}$, we let

$$(2.54) \quad V_n = \left\{ x : |D|_{\mathcal{X}(\mu),\text{loc}} \in (\|D\|_{\mathcal{X}(\mu)}/(n+1), \|D\|_{\mathcal{X}(\mu)}/n] \right\};$$

then

$$(2.55) \quad \tilde{D} = \sum_{\substack{n=1 \\ \mu(V_n) > 0}}^{\infty} \frac{\chi_{V_n}}{\chi_{V_n} |D|_{\mathcal{X}(\mu),\text{loc}}} D$$

defines a derivation, **the normalization of D** , with $\left| \tilde{D} \right|_{\mathcal{X}(\mu),\text{loc}} = \chi_{(N_D)^c}$. We will, with slight abuse of notation, denote the normalization of D by $D/|D|_{\mathcal{X}(\mu),\text{loc}}$.

Proof. The definition of \tilde{D} by (2.55) is well-posed because $\chi_{V_n} \left| \tilde{D} \right|_{\mathcal{X}(\mu),\text{loc}}$ is invertible in $L^\infty(\mu \llcorner V_n)$ if $\mu(V_n) > 0$. Moreover, the V_n are uniquely determined up to μ -null sets and so \tilde{D} does not depend on the choice of a Borel representative for $|D|_{\mathcal{X}(\mu),\text{loc}}$. Note that for $f \in \text{Lip}_b(X)$ one has

$$(2.56) \quad \chi_{V_n} |Df| \leq \chi_{V_n} |D|_{\mathcal{X}(\mu),\text{loc}} \|f\|_{\text{Lip}_b(X)},$$

and that the sets $\{V_n : \mu(V_n) > 0\}$ are an $L^\infty(\mu \llcorner N_D^c)$ -Borel partition of unity. Thus (2.55) provides a bounded linear map $\tilde{D} : \text{Lip}_b(X) \rightarrow L^\infty(\mu)$ with norm at most 1. Note also that \tilde{D} satisfies the product rule because D does.

We show that \tilde{D} is weak* continuous; by the Krein-Šmulian Theorem, it suffices to check continuity for bounded nets. Therefore, suppose that $g \in L^1(\mu)$ and $f_\eta \xrightarrow{w^*} f$ where the set $\{f_\eta\}_\eta \cup \{f\}$ is contained in the ball of radius M in $\text{Lip}_b(X)$. For each $\varepsilon > 0$ there is an N such that for all h of norm at most M in $\text{Lip}_b(X)$,

$$(2.57) \quad \left| \sum_{\substack{n > N \\ \mu(V_n) > 0}}^{\infty} \int g \frac{\chi_{V_n}}{\chi_{V_n} |D|_{\mathcal{X}(\mu),\text{loc}}} D h \, d\mu \right| \leq \varepsilon;$$

as

$$(2.58) \quad \tilde{D}_N = \sum_{\substack{n \leq N \\ \mu(V_n) > 0}}^{\infty} \frac{\chi_{V_n}}{\chi_{V_n} |D|_{\mathcal{X}(\mu), \text{loc}}} D$$

is a derivation,

$$(2.59) \quad \lim_{\eta} \int g \tilde{D}_N f_{\eta} d\mu = \int g \tilde{D}_N f d\mu;$$

combining (2.57) and (2.59), we conclude that

$$(2.60) \quad \lim_{\eta} \int g \tilde{D} f_{\eta} d\mu = \int g \tilde{D} f d\mu,$$

which shows that \tilde{D} is weak* continuous.

We observe that χ_{N_D} annihilates \tilde{D} ; thus, to show that $\left| \tilde{D} \right|_{\mathcal{X}(\mu), \text{loc}} = \chi_{N_D^c}$, it suffices to show that if the subset $U \subset N_D^c$ has positive measure, then $\|\chi_U \tilde{D}\|_{\mathcal{X}(\mu)} = 1$. This follows because, for some n , $\mu(U \cap V_n) > 0$ and

$$(2.61) \quad \chi_{U \cap V_n} \left| \tilde{D} \right|_{\mathcal{X}(\mu), \text{loc}} = \left| \chi_{U \cap V_n} \tilde{D} \right|_{\mathcal{X}(\mu), \text{loc}} = \chi_{U \cap V_n}.$$

□

Proof of Lemma 2.50. Without loss of generality, we can assume that μ is finite. We first prove that for each $\varepsilon > 0$ there is a Borel $L^\infty(\mu)$ -partition of unity $\{V_\alpha\}$ such that:

- For each α there are 1-Lipschitz functions $\{g_{\alpha,j}\}_{j=1}^k$ and unit norm derivations $\{\tilde{D}_{\alpha,i}\}_{i=1}^k \subset \chi_{V_\alpha} \mathcal{X}(\mu)$;
- The submodule generated by the derivations $\{\tilde{D}_{\alpha,i}\}_{i=1}^k \subset \chi_{V_\alpha} \mathcal{X}(\mu)$ contains that generated by the derivations $\{\chi_{V_\alpha} D_i\}_{i=1}^k$;
- The matrix $(\chi_{V_\alpha} \tilde{D}_{\alpha,i} g_j)_{i,j=1}^k$, with entries in $L^\infty(\mu|_{V_\alpha})$, is upper triangular;
- Each entry λ on the diagonal of $(\chi_{V_\alpha} \tilde{D}_{\alpha,i} g_j)_{i,j=1}^k$ satisfies $\lambda \geq 1 - \varepsilon$ (in the ring $L^\infty(\mu|_{V_\alpha})$).

We will refer to this property as $P(k, \varepsilon)$ and it will be proved by induction on k .

For $k = 1$, we first replace D_1 by its normalization \tilde{D}_1 (Lemma 2.53) to have $\left| \tilde{D}_1 \right|_{\mathcal{X}(\mu), \text{loc}} = 1$, as D_1 is nowhere vanishing. Note that (2.55) implies that $D_1 = |D_1|_{\mathcal{X}(\mu), \text{loc}} \tilde{D}_1$. We know that the class \mathcal{C}_1 of Borel subsets W such that there is a 1-Lipschitz g with

$$(2.62) \quad D_1 g \geq 1 - \varepsilon \quad \mu\text{-a.e. on } W,$$

is not empty. We choose

$$(2.63) \quad \mu(V_1) \geq \frac{1}{2} \sup_{W \in \mathcal{C}_1} \mu(W)$$

and keep going exhausting X in μ -measure (compare the proof of Theorem 2.43 in [Sch]). The functions g_α are chosen accordingly to the sets V_α so that (2.62) holds. Then one lets $\tilde{D}_{\alpha,1} = \chi_{V_\alpha} \tilde{D}_1$. The derivation $\chi_{V_\alpha} D_1$ belongs to the submodule generated by $\tilde{D}_{\alpha,1}$ because $\chi_{V_\alpha} D_1 = |D_1|_{\mathcal{X}(\mu), \text{loc}} \tilde{D}_{\alpha,1}$.

We now show that $P(k+1, \varepsilon)$ follows from $P(k, \varepsilon)$. Using $P(k, \varepsilon)$ for the derivations $\{D_i\}_{i=1}^k$ we can assume, by replacing μ with a restriction $\mu \llcorner V$, that there are 1-Lipschitz functions $\{g_j\}_{j=1}^k$ and derivations $\{\tilde{D}_i\}_{i=1}^k$ such that $P(k, \varepsilon)$ holds. We consider the normalization \tilde{D}_{k+1} of

$$(2.64) \quad D_{k+1} - \sum_{i=1}^k \frac{D_{k+1}g_i}{\tilde{D}_i g_i} \tilde{D}_i,$$

so that

$$(2.65) \quad D_{k+1}g_j = 0 \quad (1 \leq j \leq k);$$

note that D_{k+1} belongs to the submodule generated by the derivations $\{\tilde{D}_i\}_{i=1}^{k+1}$. We now apply the argument used in the case $k = 1$ to the derivation \tilde{D}_{k+1} in order to complete the proof of $P(k+1, \varepsilon)$.

If M_α denotes the matrix $(\tilde{D}_{\alpha,i}g_{\alpha,j})_{i,j=1}^k$, its determinant satisfies the bounds:

$$(2.66) \quad (1 - \varepsilon)^k \leq \det M_\alpha \leq \beta(k),$$

where $\beta(k)$ is a univerval constant depending only on k . In particular, letting

$$(2.67) \quad D_{\alpha,i} = \sum_{j=1}^k (M_\alpha^{-1})_{i,j} \tilde{D}_{\alpha,j},$$

we have $|D_{\alpha,i}|_{\mu \llcorner V_\alpha, \text{loc}} \leq C(k, \varepsilon)$ and $D_{\alpha,i}g_{\alpha,j} = \delta_{i,j} \chi_{V_\alpha}$. Moreover, solving (2.67) for the derivations $\{\tilde{D}_{\alpha,i}\}_{i=1}^k$ shows that the derivations $\{\chi_{V_\alpha} D_i\}_{i=1}^k$ belong to the submodule generated by the $\{D_{\alpha,i}\}_{i=1}^k$. \square

Consider a Lipschitz map $F : X \rightarrow Y$ and a Radon measure μ on X ; given a derivation $D \in \mathcal{X}(\mu)$ the **push forward** $F_\# D \in \mathcal{X}(F_\# \mu)$ is the derivation defined by:

$$(2.68) \quad \int_Y g(F_\# D) f dF_\# \mu = \int_X g \circ F D(f \circ F) d\mu \quad (\forall (f, g) \in \mathcal{D}^1(Y)).$$

We now recall the notion of 1-forms which are dual to derivations.

Definition 2.69. The **module of 1-forms** $\mathcal{E}(\mu)$ is the dual module of $\mathcal{X}(\mu)$, i.e. it consists of the bounded module homomorphisms $\mathcal{X}(\mu) \rightarrow L^\infty(\mu)$. The module $\mathcal{E}(\mu)$ is an $L^\infty(\mu)$ -normed module and the local norm will be denoted by $|\cdot|_{\mathcal{E}(\mu), \text{loc}}$.

To each $f \in \text{Lip}_b(X)$ one can associate the 1-form $df \in \mathcal{E}(\mu)$ by letting:

$$(2.70) \quad \langle df, D \rangle = Df \quad (\forall D \in \mathcal{X}(\mu));$$

the map $d : \text{Lip}_b(X) \rightarrow \mathcal{E}(\mu)$ is a weak* continuous 1-Lipschitz linear map satisfying the product rule $d(fg) = gdf + fdg$.

Note that because of Lemma 2.47 one can extend the domain of d to $\text{Lip}(X)$ so that if f is Lipschitz, df is a well-defined element of $\mathcal{E}(\mu)$ and $\|df\|_{\mathcal{E}(\mu)} \leq \mathbf{L}(f)$.

2.4. Correspondence between derivations and Alberti representations. In this Subsection we recall some results in [Sch13] about the correspondence between derivations and Alberti representations. Throughout this Subsection $F : X \rightarrow \mathbb{R}^k$ denotes a Lipschitz function, $\alpha \in (0, \pi/2)$ an angle, δ a positive constant, $w \in \mathbb{S}^{k-1}$ a unit vector and $\{u_i\}_{i=1}^{k-1}$ an orthonormal basis for the orthogonal complement of w .

We first recall an *approximation scheme* (Theorem 3.66 in [Sch13]) which relates Alberti representations and the weak* topology on $\text{Lip}_b(X)$:

Theorem 2.71. *Let X be a compact metric space and μ a Radon measure on X . Suppose that $K \subset X$ is compact and $\text{Frag}(X, F, \mathcal{C}(w, \alpha), \langle w, F \rangle, \geq \delta)$ -null. Denoting by $d_{\delta, \alpha}$ the distance:*

$$(2.72) \quad d_{\delta, \alpha}(x, y) = \delta d(x, y) + \cot \alpha \sum_{i=1}^{k-1} |\langle u_i, F(x) - F(y) \rangle|,$$

there is a sequence of real-valued Lipschitz functions $\{g_n\}$ and a Borel $S \subset K$ such that:

- (1) $\mu(K \setminus S) = 0$;
- (2) $g_n \xrightarrow{w^*} \langle w, F \rangle$;
- (3) *for each $x \in S$ and each n there is an $r_n > 0$ such that the restriction $g_n|_{B(x, r_n)}$ is 1-Lipschitz with respect to the distance $d_{\delta, \alpha}$.*

We will use the following consequence of Theorem 2.71 whose proof is contained in the proofs of Lemma 3.68 and Lemma 3.76 in [Sch13].

Lemma 2.73. *Let X be a complete separable metric measure space and μ a Radon measure on X . Suppose that the compact set $K \subset X$ is $\text{Frag}(X, F, \mathcal{C}(w, \alpha), \langle w, F \rangle, > \delta)$ -null. Then there are bounded Lipschitz functions $\tilde{f}_n \xrightarrow{w^*} \tilde{f}$ such that:*

- (1) *For each $x \in K$, $\tilde{f}(x) = \langle w, F(x) \rangle$;*
- (2) *For each m there are countably many disjoint Borel sets $\{S_{m, \alpha}\}$ with $\mu(K \setminus \bigcup_{\alpha} S_{m, \alpha}) = 0$;*
- (3) *For each (m, α) there is a sequence $\{\tilde{f}_{m, \alpha, n}\} \subset \text{Lip}_b(X)$ with $\tilde{f}_{m, \alpha, n} \xrightarrow{w^*} \tilde{f}_{m, \alpha}$ and $\tilde{f}_{m, \alpha} = \tilde{f}_m$ on $S_{m, \alpha}$;*
- (4) *For each (m, α, n) there are finitely many points $\{x_{m, \alpha, n, j}\} \subset S_{m, \alpha}$ and finitely many disjoint Borel sets $\{S_{m, \alpha, n, j}\}$ with $\bigcup_j S_{m, \alpha, n, j} = S_{m, \alpha}$ and such that for each $x \in S_{m, \alpha, n, j}$ one has:*

$$(2.74) \quad \tilde{f}_{m, \alpha, n}(x) = \tilde{f}_{m, \alpha}(x) + \delta d(x, x_{m, \alpha, n, j}) + \cot \alpha \sum_{i=1}^{k-1} |\langle u_i, F(x) - F(x_{m, \alpha, n, j}) \rangle|.$$

In Theorem 3.11 in [Sch13] it was shown that to a C -Lipschitz Alberti representation \mathcal{A} of the measure μ it is possible to associate a derivation $D_{\mathcal{A}} \in \mathcal{X}(\mu)$ by using the formula:

$$(2.75) \quad \int_X g D_{\mathcal{A}} f d\mu = \int_{\text{Frag}(X)} dP(\gamma) \int_{\text{dom } \gamma} (f \circ \gamma)'(t) g \circ \gamma(t) d(\gamma^{-1} \# \nu_{\gamma})(t) \quad (g \in L^1(\mu) \cap \mathcal{B}^\infty(X))$$

to define $D_{\mathcal{A}} f$; moreover, one has the norm bound $\|D_{\mathcal{A}}\|_{\mathcal{X}(\mu)} \leq C$ and if the Alberti representation \mathcal{A} is in the F -direction of the k -dimensional cone field \mathcal{C} , one has $D_{\mathcal{A}} F(x) \in \mathcal{C}(x)$ for μ -a.e. x .

In order to compare the derivations associated to different Alberti representations the following notion of independence for cone fields is useful:

Definition 2.76. We say that the n -dimensional cone fields $\{\mathcal{C}_i\}_{i=1}^k$ are **independent** if for each $x \in X$ and each choice of $v_{i,x} \in \mathcal{C}_i(x)$, the vectors $\{v_{i,x}\}_{i=1}^k$ are linearly independent.

Note that if the Alberti representations $\{\mathcal{A}_i\}_{i=1}^k$ are in the F -directions of independent cone fields, the corresponding derivations $\{D_{\mathcal{A}_i}\}_{i=1}^k$ are independent. We will use the following result (Corollary 3.94 in [Sch13]):

Corollary 2.77. *Suppose that the measure μ admits Alberti representations in the F -direction of k independent cone fields. Then for each $\varepsilon > 0$ and each k -dimensional cone field \mathcal{C} , the measure μ admits a $(1, 1 + \varepsilon)$ -Alberti representation in the F -direction of \mathcal{C} .*

A special case of the previous result is the following:

Corollary 2.78. *Suppose that the functions $\{F_i\}_{i=1}^k$ are pseudodual to the derivations $\{D_i\}_{i=1}^k$; then for any k -dimensional cone field \mathcal{C} , the measure μ admits an Alberti representation in the F -direction of \mathcal{C} .*

3. 1-DIMENSIONAL CURRENTS AND DERIVATIONS

The goal of this Section is to make precise the correspondence between 1-dimensional metric currents and derivations via Theorem 3.7.

Lemma 3.1. *Consider a metric functional $T \in \text{MF}_k(X)$ with finite mass. If $B \subset X$ is Borel and $\|T\|(B) > 0$, then for each $\eta \in (0, 1)$ there are disjoint Borel sets $B_i \subset B$ and 1-Lipschitz functions⁶ $\pi^i : X \rightarrow \mathbb{R}^k$:*

$$(3.2a) \quad \|T\| \left(B \setminus \bigcup_i B_i \right) = 0;$$

$$(3.2b) \quad |T(\chi_{B_i}, \pi_1^i, \dots, \pi_k^i)| > \eta \|T\|(B_i).$$

Proof. The proof uses [AK00, Prop. 2.7] (characterization of mass): for each $\varepsilon > 0$ there are disjoint Borel sets $B_i \subset B$ and 1-Lipschitz functions $\pi^i : X \rightarrow \mathbb{R}^k$:

$$(3.3) \quad B = \bigcup_i B_i;$$

$$(3.4) \quad \sum_i \left(\|T\|(B_i) - |T(\chi_{B_i}, \pi_1^i, \dots, \pi_k^i)| \right) < \varepsilon;$$

let $J_\alpha = \{i : |T(\chi_{B_i}, \pi_1^i, \dots, \pi_k^i)| \leq \alpha \|T\|(B_i)\}$; then one has:

$$(3.5) \quad (1 - \alpha) \sum_{i \in J_\alpha} \|T\|(B_i) < \varepsilon;$$

so

$$(3.6) \quad \|T\| \left(\bigcup_{i \in J_\alpha} B_i \right) < \frac{\varepsilon}{1 - \alpha};$$

therefore the conclusion of the Lemma is true for those $i \notin J_\alpha$ which cover all but $\frac{\varepsilon}{1 - \alpha}$ of the $\|T\|$ -measure of B . The result follows by an exhaustion argument. \square

⁶with respect to the l^∞ -norm

Theorem 3.7. *Let μ be a finite Radon measure on X . There is a map*

$$(3.8) \quad \begin{aligned} \text{Der}_\mu : \mathbf{M}_1(\mu) &\rightarrow \mathcal{X}(\mu) \\ T &\mapsto D_T \end{aligned}$$

where $D_T \in \mathcal{X}(\|T\|)$ is the unique derivation satisfying

$$(3.9a) \quad T(f, \pi) = \int f D_T \pi d\|T\| \quad (\forall (f, \pi) \in L^1(\|T\|) \times \text{Lip}(X))$$

$$(3.9b) \quad |D_T|_{\mathcal{X}(\|T\|), \text{loc}} = 1.$$

Conversely, there is an $L^\infty(\mu)$ -module homomorphism map

$$(3.10) \quad \begin{aligned} \text{Cur}_\mu : \mathcal{X}(\mu) &\rightarrow \mathbf{M}_1(\mu) \\ D &\mapsto T_D \end{aligned}$$

where T_D is the unique current satisfying

$$(3.11a) \quad T_D(f, \pi) = \int f D \pi d\mu \quad (\forall (f, \pi) \in L^1(\|T\|) \times \text{Lip}(X))$$

$$(3.11b) \quad \|T_D\| = |D|_{\mathcal{X}(\mu), \text{loc}} \mu.$$

Proof. Given $T \in \mathbf{M}_1(\mu)$, for a fixed $f \in \text{Lip}_b(X)$ one defines a linear functional on $L^1(\|T\|)$ by:

$$(3.12) \quad g \mapsto T(g, f) \quad (g \in L^1(\|T\|));$$

the Riesz Representation Theorem gives a unique $D_T f \in L^\infty(\|T\|)$:

$$(3.13) \quad \int_X g D_T f d\|T\| = T(g, f);$$

the map $D_T : \text{Lip}_b(X) \rightarrow L^\infty(\|T\|)$ is a derivation because:

- It is linear by linearity of currents;
- It is bounded with norm 1 because:

$$(3.14) \quad \left| \int_X g D_T f d\|T\| \right| \leq \mathbf{L}(f) \int_X |g| d\|T\|;$$

- The product rule follows from [AK00, Eq. 3.1 in Thm. 3.5];
- The weak* continuity follows from the continuity axiom for currents ((4) in Defn. 2.11).

Note that the module $\mathcal{X}(\|T\|)$ can be canonically identified with the submodule $\chi_{U_T} \mathcal{X}(\mu)$ where

$$(3.15) \quad U_T = \left\{ x \in X : \frac{d\|T\|}{d\mu}(x) > 0 \right\},$$

so Der_μ is well-defined.

By Lemma 3.1, for each $\eta \in (0, 1)$ we can find disjoint Borel sets B_i and 1-Lipschitz functions $\pi^i \in \text{Lip}(X)$ with $\|T\|(X \setminus \bigcup_i B_i) = 0$ and

$$(3.16) \quad T(\chi_{B_i}, \pi^i) > \eta \|T\|(B_i);$$

in particular, for each $n \in \mathbb{N}$ one has $\chi_{S_i} D_T \pi^i \geq \frac{\eta}{n+1} \eta \chi_{S_i}$, where S_i is a subset of B_i of measure at least $\frac{\eta}{n+1} \|B_i\|$; using an exhaustion argument and then letting $\eta \rightarrow 1$ and $n \nearrow \infty$, we conclude that (3.9b) holds. Note that we have used the fact that each derivation $D \in \mathcal{X}(\mu)$ can be canonically extended to a map $D : \text{Lip}(X) \rightarrow L^\infty(\mu)$ (see Remark 2.115 in [Sch13]).

We now prove the second part of this Theorem; note that for $D \in \mathcal{X}(\mu)$ (3.11a) uniquely determines a current $T_D \in \mathbf{M}_1(\mu)$ because the axioms of metric currents follow from the corresponding properties of derivations. Note also that $T_{D_1+D_2} = T_{D_1} + T_{D_2}$ and $T_{\lambda D} = T_D \llcorner \lambda$, showing that Cur_μ is an $L^\infty(\mu)$ -module homomorphism.

As $|D\pi| \leq \mathbf{L}(\pi) |D|_{\mathcal{X}(\mu), \text{loc}}$, $\|T_D\| \leq |D|_{\mathcal{X}(\mu), \text{loc}} \mu$. On the other hand, for each $\eta \in (0, 1)$ and each Borel set A , we can find disjoint Borel sets $B_i \subset A$ and 1-Lipschitz functions π^i with $\|T\|(A \setminus \bigcup_i B_i) = 0$ and

$$(3.17) \quad \chi_{B_i} D \pi^i \geq \eta \chi_{B_i} |D|_{\mathcal{X}(\mu), \text{loc}};$$

in particular,

$$(3.18) \quad \|T_D\|(A) \geq \eta \int_A |D|_{\mathcal{X}(\mu), \text{loc}} d\mu$$

which implies (3.11b). \square

Remark 3.19. From Theorem 3.7 one obtains the following identities:

$$(3.20) \quad \text{Cur}_\mu(\text{Der}_\mu(T)) \llcorner \frac{d\|T\|}{d\mu} = T$$

$$(3.21) \quad \text{Der}_\mu(\text{Cur}_\mu(D)) = \frac{D}{|D|_{\mathcal{X}(\mu), \text{loc}}}.$$

4. CURRENTS AND ALBERTI REPRESENTATIONS

The goal of this Section is to prove Theorem 1.3. Throughout this Section we will denote by $\{e_i\}_{i=1}^k$ the standard basis of \mathbb{R}^k . In the proof of Theorem 1.3 we will use the following consequence of Rainwater's Lemma [Rai69] (compare Corollary 5.8 in [Bat12] and Lemma 2.56 in [Sch13]):

Lemma 4.1. *Let X be a complete separable metric space and μ a Radon measure on X . Let $f : X \rightarrow \mathbb{R}^k$ be a Lipschitz map, $w \in \mathbb{S}^{k-1}$, $\alpha \in (0, \pi/2)$ and $\delta > 0$. For any Borel subset $B \subset X$ there are disjoint Borel sets A, S such that:*

- (1) $A \cup S = B$;
- (2) *The measure $\mu \llcorner A$ admits an Alberti representation in the f -direction of $\mathcal{C}(w, \alpha)$ with $\langle w, f \rangle$ -speed $\geq \delta$;*
- (3) *The set S is $\text{Frag}(X, f, \mathcal{C}(w, \alpha), \langle w, f \rangle, \geq \delta)$ -null.*

The proof of Theorem 1.3 relies on the following Lemma:

Lemma 4.2. *Suppose that $T(\chi_B, \pi_1, \dots, \pi_k) \geq \eta \|T\|(B)$, where B is Borel and $\pi : X \rightarrow \mathbb{R}$ is 1-Lipschitz and $\eta > 0$; then for all pairs $(\delta, \alpha) \in (0, \eta) \times (0, \pi/2)$ there is a Borel partition $B = A_{e_i} \cup S_{e_i}$ with $\|T\| \llcorner A_{e_i}$ admitting an Alberti representation in the π -direction of $\mathcal{C}(e_i, \alpha)$ with π_i -speed $\geq \delta$ and $\|T\|(A_{e_i}) \geq (\eta - \delta) \|T\|(B)$.*

Proof. Without loss of generality, we assume $i = 1$. Because of Lemma 4.1 we will obtain an upper bound on $\mu(K)$, where $K \subset B$ is compact and $\text{Frag}(X, \pi, \mathcal{C}(e_1, \alpha), \pi_1, \geq \delta)$ -null. We apply Lemma 2.73 and we will use the notation from its statement in the remainder of the proof. In particular, we take $w = e_1$, $u_i = e_{1+i}$ and $F = (\pi_i)_{i=1}^k$.

The following estimate is obtained by using the locality axiom ((3) in Definition 2.11) and (2.74):

$$(4.3) \quad \left| T(\chi_{S_{m,\alpha,n,j}}, \tilde{f}_{m,\alpha,n}, \pi_2, \dots, \pi_k) \right| \leq \delta \left| T(\chi_{S_{m,\alpha,n,j}}, d(\cdot, x_{m,\alpha,n,j}), \pi_2, \dots, \pi_k) \right| \\ + \cot \alpha \sum_{\beta > 1} \left| T(\chi_{S_{m,\alpha,n,j}}, |\pi_\beta - \pi_\beta(x_{m,\alpha,n,j})|, \pi_2, \dots, \pi_k) \right|;$$

we now let

$$(4.4) \quad S_{m,\alpha,n,j,\beta+} = \{x \in S_{m,\alpha,n,j} : \pi_\beta(x) \geq \pi_\beta(x_{m,\alpha,n,j})\} \\ S_{m,\alpha,n,j,\beta-} = \{x \in S_{m,\alpha,n,j} : \pi_\beta(x) < \pi_\beta(x_{m,\alpha,n,j})\},$$

and conclude that, for $\beta > 1$,

$$(4.5) \quad T(\chi_{S_{m,\alpha,n,j}}, |\pi_\beta - \pi_\beta(x_{m,\alpha,n,j})|, \pi_2, \dots, \pi_k) \\ = T(\chi_{S_{m,\alpha,n,j,\beta+}}, \pi_\beta - \pi_\beta(x_{m,\alpha,n,j}), \pi_2, \dots, \pi_k) \\ - T(\chi_{S_{m,\alpha,n,j,\beta-}}, \pi_\beta - \pi_\beta(x_{m,\alpha,n,j}), \pi_2, \dots, \pi_k) \\ = T(\chi_{S_{m,\alpha,n,j,\beta+}}, \pi_\beta, \pi_2, \dots, \pi_k) \\ - T(\chi_{S_{m,\alpha,n,j,\beta-}}, \pi_\beta, \pi_2, \dots, \pi_k) \\ = 0$$

where in the last inequality we used that currents are alternating. Combining (4.3) and (4.5) we obtain:

$$(4.6) \quad \left| T(\chi_{S_{m,\alpha}}, \tilde{f}_{m,\alpha,n}, \pi_2, \dots, \pi_k) \right| \leq \delta \|T\|(S_{m,\alpha}).$$

If we let $n \nearrow \infty$ we obtain:

$$(4.7) \quad \left| T(\chi_{S_{m,\alpha}}, \tilde{f}_{m,\alpha}, \pi_2, \dots, \pi_k) \right| \leq \delta \|T\|(S_{m,\alpha});$$

but by Lemma 2.73 $\tilde{f}_{m,\alpha} = \tilde{f}_m$ on $S_{m,\alpha}$ and summing in α we conclude that:

$$(4.8) \quad \left| T(\chi_K, \tilde{f}_m, \pi_2, \dots, \pi_k) \right| \leq \delta \|K\|;$$

letting $m \nearrow \infty$ and using that $\tilde{f} = \pi_1$ on K we conclude that

$$(4.9) \quad |T(\chi_K, \pi_1, \pi_2, \dots, \pi_k)| \leq \delta \|K\|.$$

The proof is completed by applying Lemma 4.1. \square

Proof of Theorem 1.3. For $\eta \in (0, 1)$ let the sets B_j and the functions π^j satisfy the conclusion of Lemma 3.1 for $B = X$. Let $\alpha \in (0, \pi/2)$ be such that the cone fields $\{\mathcal{C}(e_i, \alpha)\}_{i=1}^k$ are independent. For $\delta > 0$, Lemma 4.2 gives a partition $B_j = A_{j,e_1} \cup S_{j,e_1}$ with $\|T\| \llcorner A_{j,e_1}$ admitting an Alberti representation in the π^j -direction of $\mathcal{C}(e_1, \alpha)$ with π_1^j -speed $\geq \delta$ and

$$(4.10) \quad \|T\|(A_{j,e_1}) \geq (\eta - \delta) \|T\|(B);$$

proceeding by induction and applying Lemma 3.1, we obtain a partition

$$(4.11) \quad B_j = A_{j,e_1,\dots,e_k} \cup S_{j,e_1,\dots,e_k}$$

with $\|T\| \llcorner A_{j,e_1,\dots,e_k}$ admitting Alberti representations in the π^j -directions of the cone fields $\{\mathcal{C}(e_i, \alpha)\}_{i=1}^k$ and

$$(4.12) \quad \|T\|(A_{j,e_1,\dots,e_k}) \geq \underbrace{\prod_{i=1}^k (\eta - i\delta)}_c \|T\|(B).$$

If $\delta \in (0, \eta/k)$, $c > 0$; as $\|T\| \llcorner A_{j,e_1,\dots,e_k}$ admits Alberti representations in the π^j -directions of k independent cone fields, the proof is completed by applying Corollary 2.77 and an exhaustion argument. \square

Corollary 4.13. *If X is a metric space with Assouad dimension $\leq Q$, then*

$$(4.14) \quad \mathbf{M}_k(X) = \{0\}$$

for $k > Q$; moreover, if $T \in \mathbf{M}_k(X)$, the module $\mathcal{X}(\|T\|)$ is finitely generated with at most Q generators.

Proof. It follows by Theorem 1.3 and by Corollary 4.6 in [Sch13]. \square

Note that a more general result, which fully exploits the alternating property of metric currents, was obtained by Züst [Züs11, Prop. 2.5] who showed that $\mathbf{M}_k(X) = \{0\}$ for k strictly larger than the Nagata dimension of the space X . The class of spaces with finite Nagata dimension is larger than the class of spaces with finite Assouad dimension and the Assouad dimension bounds the Nagata dimension from above [LR13, Thm. 1.1].

5. A REPRESENTATION FORMULA

The goal of this Section is to prove Theorem 1.4 and the representation formula (1.5) which expresses metric currents in terms of derivations. We will use some terminology and results from Subsection 7.1 where, roughly speaking, we construct the exterior powers of the modules $\mathcal{X}(\mu)$ and $\mathcal{E}(\mu)$. The dispirited reader may just want to think of expressions like $D_1 \wedge \dots \wedge D_k$ and $df_1 \wedge \dots \wedge df_k$ as analogues of measurable k -vectors and k -covectors fields and keep in mind that as $\mathcal{X}(\mu)$ and $\mathcal{E}(\mu)$ are $L^\infty(\mu)$ -normed modules, their exterior products can be constructed in three different categories: Banach spaces, $L^\infty(\mu)$ -modules and $L^\infty(\mu)$ -normed modules.

Remark 5.1. We construct a bilinear pairing between the $L^\infty(\mu)$ -normed modules $\text{Ext}_{\mu,\text{loc}}^k \mathcal{X}(\mu)$ and $\text{Ext}_{\mu,\text{loc}}^k \mathcal{E}(\mu)$; for notational simplicity, we will let $\mathcal{X}^k(\mu) = \text{Ext}_{\mu,\text{loc}}^k \mathcal{X}(\mu)$ and $\mathcal{E}^k(\mu) = \text{Ext}_{\mu,\text{loc}}^k \mathcal{E}(\mu)$. Consider the map:

$$(5.2) \quad \begin{aligned} \Phi : (\mathcal{X}(\mu))^k \times (\mathcal{E}(\mu))^k &\rightarrow L^\infty(\mu) \\ ((D_1, \dots, D_k), (\omega_1, \dots, \omega_k)) &\mapsto \det(\langle D_i, \omega_j \rangle)_{i,j=1}^k. \end{aligned}$$

For a fixed k -tuple $\Omega = (\omega_1, \dots, \omega_k)$, the map

$$(5.3) \quad \begin{aligned} \Phi_\Omega : (\mathcal{X}(\mu))^k &\rightarrow L^\infty(\mu) \\ (D_1, \dots, D_k) &\mapsto \Phi((D_1, \dots, D_k), \Omega) \end{aligned}$$

is alternating $L^\infty(\mu)$ -multilinear and satisfies the bound

$$(5.4) \quad \begin{aligned} |\Phi_\Omega(D_1, \dots, D_k)| &\leq \sum_{\sigma \in \text{Perm}(k)} \prod_{i=1}^k |\langle D_{\sigma(i)}, \omega_i \rangle| \\ &\leq k! \prod_{i=1}^k \|D_i\|_{\mathcal{X}(\mu), \text{loc}} \prod_{j=1}^k \|\omega_j\|_{\mathcal{E}(\mu), \text{loc}}. \end{aligned}$$

By the universal property of $\mathcal{X}^k(\mu)$ we obtain an $L^\infty(\mu)$ -homomorphism $\hat{\Phi}_\Omega : \mathcal{X}^k(\mu) \rightarrow L^\infty(\mu)$. Note that the map

$$(5.5) \quad \begin{aligned} \Psi : (\mathcal{E}(\mu))^k &\rightarrow (\mathcal{X}^k(\mu))' \\ \Omega &\mapsto \hat{\Phi}_\Omega \end{aligned}$$

is an alternating $L^\infty(\mu)$ -multilinear map with norm at most $k!$ (by (5.4)). By the universal property of $\mathcal{E}^k(\mu)$ we obtain a homomorphism $\hat{\Psi} : \mathcal{E}^k(\mu) \rightarrow (\mathcal{X}^k(\mu))'$ and thus an $L^\infty(\mu)$ -bilinear pairing

$$(5.6) \quad \begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{X}^k(\mu) \times \mathcal{E}^k(\mu) &\rightarrow L^\infty(\mu) \\ (\xi, \omega) &\mapsto \hat{\Psi}(\omega)(\xi), \end{aligned}$$

satisfying

$$(5.7) \quad |\langle \xi, \omega \rangle| \leq k! \|\xi\|_{\mathcal{X}^k(\mu), \text{loc}} \|\omega\|_{\mathcal{E}^k(\mu), \text{loc}}.$$

By a similar argument, we can produce a pairing working in the category of $L^\infty(\mu)$ -modules:

$$(5.8) \quad \langle \cdot, \cdot \rangle : \text{Ext}_\mu^k \mathcal{X}(\mu) \times \text{Ext}_\mu^k \mathcal{E}(\mu) \rightarrow L^\infty(\mu)$$

which is $L^\infty(\mu)$ -bilinear and satisfies:

$$(5.9) \quad \|\langle \xi, \omega \rangle\|_{L^\infty(\mu)} \leq k! \|\xi\|_{\text{Ext}_\mu^k \mathcal{X}(\mu)} \|\omega\|_{\text{Ext}_\mu^k \mathcal{E}(\mu)}.$$

Working in the category of Banach spaces we can produce a pairing

$$(5.10) \quad \langle \cdot, \cdot \rangle : \text{Ext}^k \mathcal{X}(\mu) \times \text{Ext}^k \mathcal{E}(\mu) \rightarrow L^\infty(\mu)$$

which is \mathbb{R} -bilinear and satisfies

$$(5.11) \quad \|\langle \xi, \omega \rangle\|_{L^\infty(\mu)} \leq k! \|\xi\|_{\text{Ext}^k \mathcal{X}(\mu)} \|\omega\|_{\text{Ext}^k \mathcal{E}(\mu)}.$$

Note that given $(D_1, \dots, D_k) \in (\mathcal{X}(\mu))^k$, we can regard $D_1 \wedge \dots \wedge D_k$ as either an element of $\mathcal{X}^k(\mu)$, or of $\text{Ext}_\mu^k \mathcal{X}(\mu)$ or of $\text{Ext}^k \mathcal{X}(\mu)$. In the sequel, unless specified all three possibilities are admitted. A similar observation can be applied to an expression $df_1 \wedge \dots \wedge df_k$ where $(f_1, \dots, f_k) \in (\text{Lip}(X))^k$ and to a pairing $\langle D_1 \wedge \dots \wedge D_k, df_1 \wedge \dots \wedge df_k \rangle$.

We now prove the local version of Theorem 1.4:

Lemma 5.12. *For $T \in \mathbf{M}_k(X)$, suppose that the module $\mathcal{X}(\|T\|)$ is free on the derivations $\{D_i\}_{i=1}^N$ which have pseudodual functions $\{g_i\}_{i=1}^N \subset \text{Lip}_b(X)$. Then there are $\{\lambda_a\}_{a \in \Lambda_{k,N}} \subset L^\infty(\|T\|)$ such that:*

$$(5.13) \quad T(f, \pi_1, \dots, \pi_k) = \sum_{a \in \Lambda_{k,N}} \int_X f \lambda_a \langle D_{a_1} \wedge \dots \wedge D_{a_k}, d\pi_1 \wedge \dots \wedge d\pi_k \rangle d\|T\|.$$

Proof. Without loss of generality we can assume that $f \in \text{Lip}_b(X)$, $|f| \leq 1$ and each π_i is 1-Lipschitz. Let $\omega = (f, \pi_1, \dots, \pi_{k-1})$ so that the current $T \llcorner \omega \in \mathbf{M}_1(X)$ satisfies $\|T \llcorner \omega\| \ll \|T\|$ by [AK00, Eq. 2.5]. By Theorem 3.7 we have:

$$(5.14) \quad T(f, \pi_1, \dots, \pi_k) = T \llcorner \omega(\pi_k) = \int_X D_{T \llcorner \omega} \pi_k d\|T\|$$

where $D_{T \llcorner \omega} = \text{Der}_{\|T\|}(T \llcorner \omega)$ is the derivation associated to the 1-dimensional current $T \llcorner \omega$.

By assumption there are bounded Borel functions $\{\lambda_i\}_{i=1}^N \subset \mathcal{B}^\infty(X)$:

$$(5.15) \quad D_{T \llcorner \omega} = \sum_{i=1}^N \lambda_i D_i.$$

Therefore for $\varepsilon > 0$ there is $\delta > 0$ such that if for some Borel set S and real numbers c_j one has

$$(5.16) \quad \max_{i=1, \dots, N} \left| D_i \left(\pi_k - \sum_{j=1}^N c_j g_j \right) \right| < \delta \quad (\|T\| \llcorner S\text{-a.e.}),$$

then

$$(5.17) \quad \left| D_{T \llcorner \omega} \left(\pi_k - \sum_{j=1}^N c_j g_j \right) \right| < \varepsilon \quad (\|T\| \llcorner S\text{-a.e.})$$

holds. From Lusin's Theorem, after choosing Borel representatives for the $\{D_j \pi_k\}_{j=1}^N$, we find disjoint Borel sets $F_1, \dots, F_p \subset \text{spt } T$:

$$(5.18) \quad \|T\|(\text{spt } T \setminus \bigcup_{\alpha=1}^p F_\alpha) < \varepsilon$$

and points $x_\alpha \in F_\alpha$:

$$(5.19) \quad \chi_{F_\alpha} |D_j \pi_k - D_j \pi_k(x_\alpha)| < \delta \quad (\forall j \in \{1, \dots, N\}).$$

Then

$$(5.20) \quad \begin{aligned} T(f, \pi_1, \dots, \pi_k) &= \int_{\text{spt } T \setminus \bigcup_{\alpha=1}^p F_\alpha} D_{T \llcorner \omega} \pi_k d\|T\| \\ &+ \sum_{\alpha=1}^p \int_{F_\alpha} D_{T \llcorner \omega} \left(\pi_k - \sum_{j=1}^N D_j \pi_k(x_\alpha) g_j \right) d\|T\| \\ &+ \sum_{j=1}^N \int_X \left(\sum_{\alpha=1}^p \chi_{F_\alpha} D_j \pi_k(x_\alpha) \right) D_{T \llcorner \omega} g_j d\|T\|; \end{aligned}$$

from which it follows (by the choice of normalization $D_{T \llcorner \omega}$ has norm ≤ 1):

$$(5.21) \quad \left| T(f, \pi_1, \dots, \pi_k) - \sum_{j=1}^N \int_X \left(\sum_{\alpha=1}^p \chi_{F_\alpha} D_j \pi_k(x_\alpha) \right) D_{T \llcorner \omega} g_j d\|T\| \right| < (1 + \|T\|(\text{spt } T))\varepsilon.$$

On the other hand,

$$\begin{aligned}
 (5.22) \quad \sum_{j=1}^N \int_X \left(\sum_{\alpha=1}^p \chi_{F_\alpha} D_j \pi_k(x_\alpha) \right) D_{T \lfloor \omega} g_j d\|T\| &= \sum_{j=1}^N \int_X D_j \pi_k D_{T \lfloor \omega} g_j d\|T\| \\
 &\quad - \sum_{j=1}^N \int_{\text{spt } T \setminus \bigcup_{\alpha=1}^p F_\alpha} D_j \pi_k D_{T \lfloor \omega} g_j d\|T\| \\
 &\quad - \sum_{j=1}^N \sum_{\alpha=1}^p \int_{F_\alpha} (D_j \pi_k - D_j \pi_k(x_\alpha)) D_{T \lfloor \omega} g_j d\|T\|;
 \end{aligned}$$

observing that:

$$\begin{aligned}
 (5.23) \quad &\left| \sum_{j=1}^N \int_{\text{spt } T \setminus \bigcup_{\alpha=1}^p F_\alpha} D_j \pi_k D_{T \lfloor \omega} g_j d\|T\| \right| \leq N\varepsilon \cdot \max_{j=1, \dots, N} \|D_j\| \cdot \max_{j=1, \dots, N} \mathbf{L}(g_j) \\
 (5.24) \quad &\left| \sum_{j=1}^N \sum_{\alpha=1}^p \int_{F_\alpha} (D_j \pi_k - D_j \pi_k(x_\alpha)) D_{T \lfloor \omega} g_j d\|T\| \right| \leq N\delta \cdot \max_{j=1, \dots, N} \mathbf{L}(g_j) \cdot \|T\|(\text{spt } T);
 \end{aligned}$$

letting $\varepsilon \rightarrow 0$, $\delta \rightarrow 0$ one concludes that

$$(5.25) \quad T(f, \pi_1, \dots, \pi_k) = \sum_{j=1}^N T(f D_j \pi_k, \pi_1, \dots, \pi_{k-1}, g_j).$$

If $\Lambda'_{k,N}$ denotes the set of k -tuples on $\{1, \dots, N\}$, by using induction in (5.25),

$$(5.26) \quad T(f, \pi_1, \dots, \pi_k) = \sum_{a \in \Lambda'_{k,N}} T(f D_{a_1} \pi_1 \cdots D_{a_k} \pi_k, g_{a_1}, \dots, g_{a_k});$$

as currents are alternating

$$(5.27) \quad T(f, \pi_1, \dots, \pi_k) = \sum_{a \in \Lambda_{k,N}} T(f \langle D_{a_1} \wedge \cdots \wedge D_{a_k}, d\pi_1 \wedge \cdots \wedge d\pi_k \rangle, g_{a_1}, \dots, g_{a_k});$$

the maps $\psi \in L^1(\|T\|) \mapsto T(\psi, g_{a_1}, \dots, g_{a_k})$ defines a linear functional on $L^1(\|T\|)$ which is represented by some $\lambda_a \in L^\infty(\|T\|)$ by the Riesz representation Theorem.

We conclude that:

$$(5.28) \quad T(f, \pi_1, \dots, \pi_k) = \sum_{a \in \Lambda_{k,N}} \int_X f \lambda_a \langle D_{a_1} \wedge \cdots \wedge D_{a_k}, d\pi_1 \wedge \cdots \wedge d\pi_k \rangle d\|T\|.$$

□

We now prove Theorem 1.4:

Proof of Theorem 1.4. Suppose that $\mathcal{X}(\|T\|)$ has N generators; then by Theorem 2.48 there is an $L^\infty(\|T\|)$ -Borel partition of unity $\{U_\beta\}_{\beta \in J}$ such that J is finite with at most N elements and $\mathcal{X}(\|T\| \lfloor U_\beta)$ is free of rank $N_\beta \leq N$. Having selected a local basis of derivations for each U_β , we can apply Lemma 2.50 to obtain an $L^\infty(\|T\|)$ -Borel partition of unity $\{V_\alpha\}$ such that:

- The module $\mathcal{X}(\|T\| \lfloor V_\alpha)$ has a basis $\{D_{\alpha,i}\}_{i=1}^{N_\alpha}$.

- The norms of the derivations $\{D_{\alpha,i}\}_{i=1}^{N_\alpha}$ are bounded by a universal constant $C(N)$.
- There are 1-Lipschitz functions $\{g_{\alpha,j}\}_{j=1}^{N_\alpha}$ pseudodual to the derivations $\{D_{\alpha,i}\}_{i=1}^{N_\alpha}$ on V_α .

The hypotheses of Lemma 5.12 are met by the currents $\{T \llcorner V_\alpha\}$ and we have local representations:

$$(5.29) \quad T \llcorner V_\alpha(f, \pi_1, \dots, \pi_k) = \sum_{a \in \Lambda_{k,N_\alpha}} \int_{V_\alpha} f \lambda_{\alpha,a} \langle D_{\alpha,a_1} \wedge \dots \wedge D_{\alpha,a_k}, d\pi_1 \wedge \dots \wedge d\pi_k \rangle d\|T\|;$$

for any subset $W \subset V_\alpha$ and any index $a \in \Lambda_{k,N_\alpha}$, letting $\pi_i = g_{\alpha,a_i}$, we obtain from (5.29) the lower bound

$$(5.30) \quad \|T\|(W) \geq T \llcorner V_\alpha(\chi_W, g_{\alpha,a_1}, \dots, g_{\alpha,a_k}) = \int_W \lambda_{\alpha,a} d\|T\|,$$

which implies the upper bound $\|\lambda_{\alpha,a}\|_{L^\infty(\|T\| \llcorner V_\alpha)} \leq 1$.

Note that we can regard Λ_{k,N_α} as a subset of $\Lambda_{k,N}$ and for $a_i \in \{1, \dots, N\} \setminus \{1, \dots, N_\alpha\}$ we will improperly use the notation D_{α,a_i} to denote the trivial derivations. Note that

$$(5.31) \quad D_{a_1} = \sum_{\alpha} \lambda_{\alpha,a_1} \chi_{V_\alpha} D_{\alpha,a_1},$$

$$(5.32) \quad D_{a_i} = \sum_{\alpha} \chi_{V_\alpha} D_{\alpha,a_i}, \quad (1 < i \leq k)$$

define elements of $\mathcal{X}(\|T\|)$ with norm bounded by $C(N)$. Therefore

$$(5.33) \quad \omega_T = \sum_{a \in \Lambda_{k,N}} D_{a_1} \wedge \dots \wedge D_{a_k}$$

defines an element of $\mathcal{X}^k(\|T\|)$ with norm at most $(C(N))^k \binom{N}{k}$. By Remark 5.1 one can also regard ω_T as an element of either $\text{Ext}_{\|T\|}^k \mathcal{X}(\|T\|)$ or $\text{Ext}^k \mathcal{X}(\|T\|)$.

We now observe that:

$$(5.34) \quad \begin{aligned} T(f, \pi_1, \dots, \pi_k) &= \sum_{\alpha} (T \llcorner V_\alpha)(f, \pi_1, \dots, \pi_k) \\ &= \sum_{\alpha} \sum_{a \in \Lambda_{k,N_\alpha}} \int_{V_\alpha} f \lambda_{\alpha,a} \langle D_{\alpha,a_1} \wedge \dots \wedge D_{\alpha,a_k}, d\pi_1 \wedge \dots \wedge d\pi_k \rangle d\|T\| \\ &= \sum_{\alpha} \sum_{a \in \Lambda_{k,N}} \int_{V_\alpha} f \langle D_{a_1} \wedge \dots \wedge D_{a_k}, d\pi_1 \wedge \dots \wedge d\pi_k \rangle d\|T\| \\ &= \sum_{\alpha} \int_X f \chi_{V_\alpha} \langle \omega_T, d\pi_1 \wedge \dots \wedge d\pi_k \rangle d\|T\| \\ &= \int_X f \langle \omega_T, d\pi_1 \wedge \dots \wedge d\pi_k \rangle d\|T\|, \end{aligned}$$

which proves (1.5). \square

Remark 5.35. A consequence of Theorem 1.4 is that one can regard a k -dimensional metric current T as a map defined on $L^1(\|T\|) \times \mathcal{E}^k(\|T\|)$. Moreover, noting that if

$T \in \mathbf{M}_k(\mu)$ one can regard $L^\infty(\|T\|)$ (respectively $\mathcal{X}^k(\|T\|)$, $\mathcal{E}^k(\|T\|)$) as submodules of $L^\infty(\mu)$ (respectively $\mathcal{X}^k(\mu)$, $\mathcal{E}^k(\mu)$), the current T can be viewed as a map defined on $L^\infty(\mu) \times \mathcal{E}^k(\mu)$ and one can take $\omega_T \in \mathcal{X}^k(\mu)$.

Remark 5.36. Note that Theorem 1.4 implies [Wil10, Thm. 1.3]. In fact, if (X, μ) is a differentiability space, by Lemma 4.1 in [Sch13] the module $\mathcal{X}(\mu)$ can be identified with the set of bounded measurable sections of the Cheeger's measurable tangent bundle $T_\mu X$ (defined in [Che99, pg. 463]). Then the module $\mathcal{X}^k(\mu)$ coincides with the set of bounded measurable sections of the k -th exterior power of $T_\mu X$; in this way, we recover [Wil10, Thm. 1.3].

For $k \geq 2$, it is not clear how to identify the elements of $\mathcal{X}^k(\mu)$ which give rise to currents. However, we have a partial result concerning normal currents. We start by generalizing the notion of *precurrents* which was introduced by Williams in the context of differentiability spaces.

Definition 5.37. Suppose that μ is a finite Radon measure on X . Then each $\xi \in \mathcal{X}^k(\mu)$ defines a k -metric functional T_ξ by:

$$(5.38) \quad T_\xi(f, \pi_1, \dots, \pi_k) = \int_X f \langle \xi, d\pi_1 \wedge \dots \wedge d\pi_k \rangle d\mu;$$

moreover, T_ξ is multilinear in the arguments (f, π_1, \dots, π_k) and alternating in the arguments (π_1, \dots, π_k) . Note also that (5.7) implies that T_ξ has finite mass:

$$(5.39) \quad \|T_\xi\| \leq k! |\xi|_{\mathcal{X}^k(\mu), \text{loc}} \mu.$$

We also have that T_ξ is local in the sense that if

$$(5.40) \quad \left\{ x : |\xi|_{\mathcal{X}^k(\mu), \text{loc}}(x) \neq 0 \right\} \subset \bigcup_{\alpha=1}^k V_\alpha,$$

where the V_α are Borel sets with π_α constant on V_α , then

$$(5.41) \quad T_\xi(f, \pi_1, \dots, \pi_k) = 0.$$

In fact, by Theorem 7.54, for each $\varepsilon > 0$ we can find $\xi' \in \mathcal{X}^k(\mu)$ of the form

$$(5.42) \quad \xi' = \sum_{i \in I_\xi} D_{i_1} \wedge \dots \wedge D_{i_k}$$

with $\|\xi - \xi'\|_{\mathcal{X}^k(\mu)} \leq \varepsilon$. Then (5.41) follows because for each $D \in \mathcal{X}(\mu)$, $\chi_{V_\alpha} D \pi_\alpha = 0$.

We will call T_ξ the **k -precurrent associated to ξ** and we will denote by $\mathbf{P}_k(\mu)$ the set of k -precurrents.

Theorem 5.43. *Given $\xi \in \mathcal{X}^k(\mu)$, if the metric functional ∂T_ξ has finite mass, then T_ξ is a normal current. If $\mathcal{X}(\mu)$ is finitely generated, the set $\mathbf{N}_k(\mu)$, which consists of the normal currents whose mass is absolutely continuous with respect to μ , coincides with the set of those $T_\xi \in \mathbf{P}_k(\mu)$ whose boundary ∂T_ξ has finite mass.*

Proof of Theorem 5.43. Assume that the metric functional ∂T_ξ has finite mass. In order to show that T_ξ is a metric current, it suffices to check the continuity axiom (4) in Definition 2.11. Suppose that $f_h \xrightarrow{w^*} f$ and $\pi_{i,h} \xrightarrow{w^*} \pi_i$ for all $1 \leq i \leq k$. Note that:

$$(5.44) \quad |T_\xi(f_h, \pi_{1,h}, \dots, \pi_{k,h}) - T_\xi(f, \pi_{1,h}, \dots, \pi_{k,h})| \leq \prod_{i=1}^k \mathbf{L}(\pi_{i,h}) \int_X |f_h - f| d\|T_\xi\|$$

so that

$$(5.45) \quad \lim_{h \rightarrow \infty} |T_\xi(f_h, \pi_{1,h}, \dots, \pi_{k,h}) - T_\xi(f, \pi_{1,h}, \dots, \pi_{k,h})| = 0.$$

Moreover, we have:

$$(5.46) \quad \begin{aligned} & T_\xi(f, \pi_{1,h}, \pi_{2,h}, \dots, \pi_{k,h}) - T_\xi(f, \pi_1, \pi_{2,h}, \dots, \pi_{k,h}) \\ &= \partial T_\xi(f(\pi_{1,h} - \pi_1), \pi_{2,h}, \dots, \pi_{k,h}) \\ & - T_\xi(\pi_{1,h} - \pi_1, f, \pi_{2,h}, \dots, \pi_{k,h}); \end{aligned}$$

as

$$(5.47) \quad |\partial T_\xi(f(\pi_{1,h} - \pi_1), \pi_{2,h}, \dots, \pi_{k,h})| \leq \prod_{i=2}^k \mathbf{L}(\pi_{i,h}) \int_X |f(\pi_{1,h} - \pi_1)| d\|\partial T_\xi\|,$$

$$(5.48) \quad |T_\xi(\pi_{1,h} - \pi_1, f, \pi_{2,h}, \dots, \pi_{k,h})| \leq \mathbf{L}(f) \prod_{i=2}^k \mathbf{L}(\pi_{i,h}) \int_X |\pi_{1,h} - \pi_1| d\|T_\xi\|,$$

from (5.46) we have:

$$(5.49) \quad \lim_{h \rightarrow \infty} |T_\xi(f, \pi_{1,h}, \pi_{2,h}, \dots, \pi_{k,h}) - T_\xi(f, \pi_1, \pi_{2,h}, \dots, \pi_{k,h})| = 0.$$

Using that T_ξ is alternating in the last k arguments and induction in i , the previous argument gives:

$$(5.50) \quad \lim_{h \rightarrow \infty} |T_\xi(f_h, \pi_{1,h}, \pi_{2,h}, \dots, \pi_{k,h}) - T_\xi(f, \pi_1, \pi_2, \dots, \pi_k)| = 0,$$

which shows that T_ξ is a metric current. As ∂T_ξ has finite mass, the current T_ξ is normal. The second part of this Theorem follows from Theorem 1.4 because, if $\mathcal{X}(\mu)$ is finitely generated, any metric current is a precurrent. \square

6. APPLICATIONS

6.1. Approximation of 1-currents by Normal currents. The goal of this Subsection is to prove Theorem 1.6. We make the set theoretic assumption that the cardinality of any set is an Ulam number so that by [AK00, Lem 2.9] the masses of metric currents are concentrated on countable unions of compact sets. This assumption is not needed if we consider currents in separable Banach spaces.

Let $\text{Curves}(X)$ denote the set of Lipschitz maps from $[0, 1]$ to X topologized as a subspace of $K([0, 1] \times X)$. To each $\gamma \in \text{Curves}(X)$, one can then associate a normal current $[\gamma]$ by letting:

$$(6.1) \quad [\gamma](f d\pi) = \int_0^1 (f \circ \gamma)(t) (\pi \circ \gamma)'(t) dt \quad ((f, \pi) \in \mathcal{B}^\infty(X) \times \text{Lip}(X)).$$

Note that the mass measure of $[\gamma]$ can be bounded by:

$$(6.2) \quad \|[\gamma]\| \leq \gamma_\# (\text{md } \gamma \cdot \mathcal{L}^1 \llcorner [0, 1]).$$

Lemma 6.3. *Let Z be a Banach space and μ a σ -finite Radon measure on Z . Suppose that the derivations $\{D_i\}_{i=1}^k \subset \mathcal{X}(\mu)$ are independent. Then there are a Borel $L^\infty(\mu)$ -partition of unity V_α and there are, for each α , derivations $\{D_{\alpha,i}\}_{i=1}^k \subset \chi_{V_\alpha} \mathcal{X}(\mu)$ and unit norm functionals $\{z_{\alpha,j}^*\}_{j=1}^k \subset Z^*$ such that:*

- (1) *The submodule of $\mathcal{X}(\mu)$ generated by the derivations $\{D_{\alpha,i}\}_{i=1}^k$ is the same as the submodule generated by the derivations $\{\chi_{V_\alpha} D_i\}_{i=1}^k$;*

- (2) The functionals $\{z_{\alpha,j}^*\}_{j=1}^k$ are pseudodual to the derivations $\{D_{\alpha,i}\}_{i=1}^k$ on V_α .

Proof. Note that μ is concentrated on a K_σ -set, i.e. a countable union of compact sets; in particular, $\text{spt } \mu$ is separable and we can assume that Z is separable by taking the closure of the linear span of $\text{spt } \mu$. Up to passing to a Borel $L^\infty(\mu)$ -partition of unity we can assume that Z is also bounded. Let $\{z_i\} \subset Z$ be a countable dense set and for $i \neq j$ choose a unit norm linear functional $z_{i,j}^*$ with $\langle z_{i,j}^*, z_i - z_j \rangle = \|z_i - z_j\|_Z$. By the Stone-Weierstrass Theorem for Lipschitz Algebras [Wea99, Cor. 4.1.9], the family of functionals $\{z_{i,j}^*\}_{i,j}$ is a countable generating set⁷ for $\text{Lip}_b(Z)$. By [Sch, Prop. 2.35] we can find a Borel $L^\infty(\mu)$ -partition of unity $\{V_\alpha\}$ and for each α unit functionals $\{z_{\alpha,j}^*\}_{j=1}^k$ such that, letting $M_\alpha = (D_i z_{\alpha,j}^*)_{i,j=1}^k$, we have $\det M_\alpha \neq 0$ on V_α . Up to passing to a further Borel partition we can assume that for each α there is a $\delta_\alpha > 0$ such that:

$$(6.4) \quad |\det M_\alpha(x)| \in (\delta_\alpha, 2\delta_\alpha) \quad (\forall x \in V_\alpha);$$

we then let $D_{\alpha,i} = \sum_{j=1}^k (M_\alpha^{-1})_{i,j} D_j$. \square

Proof of Theorem 1.6. We make the following preliminary Observation (**Obs1**): suppose that $\sum_k T_k$ is either a finite sum of 1-currents or a series with

$$(6.5) \quad \sum_k \|T_k\|_{\mathbf{M}_1(Z)} < \infty,$$

and suppose also that for each n there is a sequence of normal currents $\{N_{k,n}\} \subset \mathbf{N}_1(Z)$ such that

$$(6.6) \quad \lim_{n \rightarrow \infty} \|T_k - N_{k,n}\|_{\mathbf{M}_1(Z)} = 0;$$

then, if we let $T = \sum_k T_k$, there is a sequence of normal currents $\{N_n\} \subset \mathbf{N}_1(Z)$ such that (1.7) holds.

As $\mathcal{X}(\|T\|)$ is finitely generated, by Theorem 2.48 and (**Obs1**) we can reduce to the case in which $\mathcal{X}(\|T\|)$ is free of rank N . Applying Lemma 6.3 and (**Obs1**), we can assume that $\mathcal{X}(\|T\|)$ has a basis consisting of derivations $\{D_i\}_{i=1}^N$ such that there are unit norm linear functionals $\{z_j^*\}_{j=1}^N$ which are pseudodual to the $\{D_i\}_{i=1}^N$. Let $z^* = (z_j^*)_{j=1}^N$ and $\{e_i\}_{i=1}^N$ the standard basis of \mathbb{R}^N ; by Corollary 2.78 for any $\alpha \in (0, \pi/2)$ the measure $\|T\|$ admits C -Lipschitz Alberti representations $\{\mathcal{A}_i\}_{i=1}^N$ with \mathcal{A}_i in the z^* -direction of $\mathcal{C}(e_i, \alpha)$ (and with positive z_i^* -speed); note that, up to taking an $L^\infty(\|T\|)$ -partition of unity and choosing α sufficiently small, we can assume that the derivations $\{D_{\mathcal{A}_i}\}_{i=1}^N$ form a basis of $\mathcal{X}(\|T\|)$. Applying Theorem 7.97, we can assume that $\mathcal{A}_i = (P_i, \nu_i)$ with $\text{spt } P_i \subset \text{Curves}(Z)$ and $(\nu_i)_\gamma = h_i \Psi_\gamma$, where h_i is a Borel function on Z and $\Psi_\gamma = \gamma_\# \mathcal{L}^1 \llcorner [0, 1]$. Denoting the derivation $\text{Der}_{\|T\|}(T)$ by D_T , there are bounded Borel functions $\{\lambda_i\}_{i=1}^N$ such that $D_T = \sum_{i=1}^N \lambda_i D_i$; but this implies that

$$(6.7) \quad T = \sum_{i=1}^N \text{Cur}_{\|T\|}(\lambda_i D_i),$$

⁷i.e. for each $f \in \text{Lip}_b(Z)$ there is a sequence of polynomials $\{P_n\} \subset \text{Lip}_b(Z)$ in the $z_{i,j}^*$ with $P_n \xrightarrow{w^*} f$.

and by **(Obs1)** we reduce to the case in which $T = \lambda D_{\mathcal{A}}$ where λ is a bounded Borel function and $\mathcal{A} = (P, \nu)$ is a C -Lipschitz Alberti representation with $\text{spt } P \subset \text{Curves}(Z)$ and $\nu_{\gamma} = h\Psi_{\gamma}$. Let μ denote the measure

$$(6.8) \quad \mu = \int_{\text{Curves}(Z)} \Psi_{\gamma};$$

note that $\|T\| \ll \mu$ and $h\lambda \in L^1(\mu)$; as $\text{Lip}_b(Z)$ is dense in $L^1(\mu)$, we can find, for each $\varepsilon > 0$, a function $g \in \text{Lip}_b(Z)$ such that:

$$(6.9) \quad \|g - h\lambda\|_{L^1(\mu)} \leq \varepsilon.$$

Note that the metric current N defined by

$$(6.10) \quad \begin{aligned} N(f d\pi) &= \int_{\text{Curves}(Z)} dP(\gamma) \int_{\gamma} f \partial_{\gamma} \pi d\Psi_{\gamma} \\ &= \int_{\text{Curves}(Z)} dP(\gamma) \int_{[0,1]} f \circ \gamma(t) (\pi \circ \gamma)'(t) dt \end{aligned}$$

is normal and so $N \ll \mu$ is normal. However, (6.9) implies that

$$(6.11) \quad \|N \llcorner g - T\|_{\mathbf{M}_1(Z)} \leq C \|g - h\lambda\|_{L^1(\mu)} \leq C\varepsilon.$$

□

6.2. Alberti representations with constant directions. In this Subsection we illustrate a different method to produce Alberti representations. This method allows to refine the way in which the direction is specified. In fact, the cone field is replaced by a vector field and one can also use countably many Lipschitz functions. This method relies on results of [PS12, PS13] on the structure of 1-dimensional normal currents.

We state the Paolini-Stepanov decomposition of normal currents using parametrized curves: note, however, that in [PS13] the result is stated using non-parametrized curves. Recall also that the metric space X is assumed Polish.

Theorem 6.12 (Corollary 3.3 in [PS13]). *Let N be a 1-dimensional normal current defined on X ; then there is a finite Radon measure η on the space $K([0, 1] \times X)$ of compact subsets of $[0, 1] \times X$ which is concentrated on $\text{Curves}(X)$, and such that:*

$$(6.13) \quad N = \int_{\text{Curves}(X)} [\gamma] d\eta(\gamma);$$

$$(6.14) \quad \|N\| = \int_{\text{Curves}(X)} \|[\gamma]\| d\eta(\gamma);$$

$$(6.15) \quad \|N\|(X) = \int_{\text{Curves}(X)} l(\gamma) d\eta(\gamma),$$

where $l(\gamma)$ denotes the length of γ which is given by:

$$(6.16) \quad l(\gamma) = \int_0^1 |\dot{\gamma}(t)| dt.$$

Note that the integrals in (6.13) and (6.14) make sense because the maps $\gamma \mapsto [\gamma]$ and $\gamma \mapsto \|[\gamma]\|$ are Borel in the following sense: for each $(f, \pi) \in \mathcal{B}^{\infty}(X) \times \text{Lip}(X)$ and each Borel $E \subset X$, the maps $\gamma \mapsto [\gamma](f d\pi)$ and $\gamma \mapsto \|[\gamma]\|(E)$ are Borel. We need to introduce more terminology:

Definition 6.17. The set of maps $\gamma \in \text{Curves}(X)$ with Lipschitz constant at most n is a Polish space and is denoted by $\text{Curves}_n(X)$. The set of Lipschitz maps $\gamma : K \rightarrow X$, where K is a nonempty compact subset of $[0, 1]$, is denoted by $\text{Pieces}(X)$ and topologized as a subset of $K([0, 1] \times X)$. Note that $\text{Pieces}(X)$ is a subset of $\text{Frag}(X)$ and a Borel subset of $K([0, 1] \times X)$. The subset of maps $\gamma \in \text{Pieces}(X)$ with Lipschitz constant at most n is a Polish space and is denoted by $\text{Pieces}_n(X)$. If $(\gamma, \tilde{\gamma}) \in \text{Curves}(X) \times \text{Pieces}(X)$ and $\gamma|_{\text{dom } \tilde{\gamma}} = \tilde{\gamma}$, we say that $\tilde{\gamma}$ is a **piece** of γ .

To each $\gamma \in \text{Pieces}(X)$, one can associate a metric current $[\gamma]$ by letting:

$$(6.18) \quad [\gamma](f d\pi) = \int_{\text{dom } \gamma} (f \circ \gamma)(t)(\pi \circ \gamma)'(t) dt \quad ((f, \pi) \in \mathcal{B}^\infty(X) \times \text{Lip}(X));$$

a modification of the argument in Lemma 3.1 in [Sch13] shows that, for each $(f, \pi) \in \mathcal{B}^\infty(X) \times \text{Lip}(X)$, the map

$$(6.19) \quad \begin{aligned} \text{Pieces}(X) &\rightarrow \mathbb{R} \\ \gamma &\mapsto [\gamma](f d\pi) \end{aligned}$$

is Borel. Having fixed an open set $U \subset X$, there is a countable collection \mathcal{F}_U of 1-forms $\omega = \sum_i f_i d\pi_i$ such that, for each $\gamma \in \text{Pieces}(X)$,

$$(6.20) \quad \|\gamma\|(U) = \sup_{\omega \in \mathcal{F}_U} [\gamma](\omega);$$

this implies that, for each Borel $E \subset X$, the map:

$$(6.21) \quad \begin{aligned} \text{Pieces}(X) &\rightarrow [0, \infty) \\ \gamma &\mapsto \|[\gamma]\|(E) \end{aligned}$$

is Borel. Note also that the mass of the current associated to $\gamma \in \text{Pieces}(X)$ can be bounded from above similarly as in (6.2):

$$(6.22) \quad \|[\gamma]\| \leq \gamma_{\#} (\text{md } \gamma \cdot \mathcal{L}^1 \llcorner \text{dom } \gamma).$$

We now discuss the notion of Alberti representations in the direction of a vector field v . In greater generality, we consider l^2 -valued Lipschitz maps, where l^2 is the Hilbert space of l^2 -summable sequences. In the following, we let \mathbb{R}^∞ denote the product of countably many copies of \mathbb{R} with the product topology. Note that any map $F : X \rightarrow l^2$ is determined by its components F_i ; in particular, if F is Lipschitz and $D \in \mathcal{X}(\mu)$, we can choose a Borel representative of each DF_i and denote by DF the Borel map $DF : X \rightarrow \mathbb{R}^\infty$ whose i -th component is DF_i . Moreover, we can stipulate that the maps $DF_i : X \rightarrow \mathbb{R}$ are uniformly bounded, with the bound independent of i . In the following, this will always be assumed when we apply a derivation $D \in \mathcal{X}(\mu)$ to a Lipschitz function $F : X \rightarrow l^2$. We finally call a Borel map $v : X \rightarrow \mathbb{R}^\infty$, such that the components v_i are uniformly bounded by some $C > 0$, a **vector field**.

Definition 6.23. Let $F : X \rightarrow l^2$ be Lipschitz and $v : X \rightarrow \mathbb{R}^\infty$ a vector field. Denote by N_v the set where v vanishes:

$$(6.24) \quad N_v = \{x \in X : v(x) = 0\}.$$

We say that the Alberti representation $\mathcal{A} = (P, \nu)$ of $\mu \llcorner (X \setminus N_v)$ is in the F -direction of v if for P -a.e. γ and \mathcal{L}^1 -a.e. $t \in \text{dom } \gamma$ there is a $\lambda = \lambda(\gamma, t) > 0$ such

that:

$$(6.25) \quad (F \circ \gamma)'(t) = \lambda v(\gamma(t)).$$

Given a Lipschitz map $F : X \rightarrow l^2$, to produce vector fields v with $\mu \llcorner (X \setminus N_v)$ admitting an Alberti representation in the F -direction of v , we will use a special class of derivations.

Definition 6.26. A derivation $D \in \mathcal{X}(\mu)$ is called **normal** if there is a Borel $L^\infty(\mu \llcorner (X \setminus N_D))$ -partition of unity $\{U_\alpha\}$ such that for each α there are:

- (1) An isometric embedding $\iota_\alpha : U_\alpha \rightarrow Z_\alpha$ where Z_α is a Polish space.
- (2) A normal current N_α in Z_α with $\iota_{\alpha\#}(\mu \llcorner U_\alpha) \ll \|N_\alpha\|$.
- (3) Denoting by $D_N \in \mathcal{X}(\|N_\alpha\|)$ the derivation associated to N_α given by Theorem 3.7, there is $\lambda_\alpha \in L^\infty(\|N_\alpha\|)$ with $\lambda_\alpha \geq 0$ and

$$(6.27) \quad \iota_{\alpha\#} \chi_{U_\alpha} D = \lambda_\alpha D_{N_\alpha}.$$

Note that in (6.27) we have used that (2) allows to identify $\iota_{\alpha\#} D$ with a derivation in $\mathcal{X}(\|N_\alpha\|)$.

Remark 6.28. We want to remark that there are many normal derivations. Suppose that μ admits an Alberti representation in the f -direction of an n -dimensional cone field \mathcal{C} . The proof of Theorem 2.64 in [Sch13] allows us to assume that there is an $L^\infty(\mu)$ -partition of unity $\{K_\alpha\}$ such that, for each α :

- (1) The set K_α is compact and embeds isometrically in S_α , which is a convex compact subset of some Banach space;
- (2) Regarding $\mu \llcorner K_\alpha$ as a measure on S_α , it admits a 1-Lipschitz Alberti representation \mathcal{A}_α in the f -direction of \mathcal{C} ;
- (3) The Alberti representation \mathcal{A}_α is of the form

$$(6.29) \quad \mu \llcorner K_\alpha = \int_{\text{Frag}(S_\alpha)} g_\alpha \Psi_\gamma dP_\alpha;$$

- (4) g_α is a bounded Borel function vanishing on $S_\alpha \setminus K_\alpha$;
- (5) The probability measure P_α is concentrated on the set $\text{Lip}_1([0, \tau_\alpha], S_\alpha)$ of 1-Lipschitz maps $[0, \tau_\alpha] \rightarrow S_\alpha$, where $\tau_\alpha \in (0, 1]$;
- (6) $\Psi_\gamma = \gamma_\# \mathcal{L}^1 \llcorner [0, \tau_\alpha]$.

We can then define a normal current $N_\alpha \in \mathbf{N}_1(S_\alpha)$ by:

$$(6.30) \quad N_\alpha = \int_{\text{Frag}(S_\alpha)} [\gamma] dP_\alpha,$$

so that $\mu \llcorner K_\alpha \ll \|N_\alpha\|$ and $D_{\mathcal{A}_\alpha} = \chi_{\{g_\alpha \neq 0\}} D_{N_\alpha}$ for some nonnegative $\lambda_\alpha \in \mathcal{B}^\infty(S_\alpha)$ which vanishes on $S_\alpha \setminus K_\alpha$. Thus, the derivation $D \in \mathcal{X}(\mu)$ defined by $D = \sum_\alpha \chi_{K_\alpha} D_{\mathcal{A}_\alpha}$ is a normal derivation. Moreover, if $\mathcal{X}(\mu)$ is finitely generated, by choosing Alberti representations in the directions of independent cone fields, we get a generating set for $\mathcal{X}(\mu)$ consisting of normal derivations. If $\mathcal{X}(\mu)$ is not finitely generated, Theorem 3.96 in [Sch13] implies that the $\text{Lip}_b(X)$ -span of the set of normal derivations is weak* dense in $\mathcal{X}(\mu)$. Note that in this case it is necessary to use the $\text{Lip}_b(X)$ -span instead of the $L^\infty(\mu)$ -span. In fact, if D_1, D_2 are normal derivations and if $\lambda_1, \lambda_2 \in L^\infty(\mu)$, then $\lambda_1 D_1 + \lambda_2 D_2$ might not be a normal derivation. However, if λ_1 and λ_2 are Lipschitz⁸, then $\lambda_1 D_1 + \lambda_2 D_2$ is a normal

⁸more precisely, λ_1 and λ_2 have Lipschitz representatives

derivation because if N is a normal current and f is Lipschitz, then $N \llcorner f$ is still a normal current.

The goal of this Subsection is the proof of the following Theorem:

Theorem 6.31. *Let $F : X \rightarrow l^2$ a Lipschitz map and $D \in \mathcal{X}(\mu)$ a normal derivation. Then $\mu \llcorner (X \setminus N_{DF})$ admits a 1-Lipschitz Alberti representation in the F -direction of DF .*

The proof of Theorem 6.31 requires some preparation and part of it has been split into some intermediate Lemmas.

Lemma 6.32. *In proving Theorem 6.31 we can assume that:*

- (1) *The metric space X is a compact subset of a Polish space Z .*
- (2) *The map $F : X \rightarrow l^2$ is 1-Lipschitz and extends to a 1-Lipschitz map $F : Z \rightarrow l^2$.*
- (3) *There is a normal current $N \in \mathbf{N}_1(Z)$ with $\mu \ll \|N\|$ and $D = \lambda D_N$, where D_N is the derivation associated to N given by Theorem 3.7, and $\lambda \in L^\infty(\|N\|)$ is nonnegative.*
- (4) *There are constants $0 < C_1 \leq C_2$ such that:*

$$(6.33) \quad C_1 \leq \frac{d\mu}{d\|N\|}(x) \leq C_2 \quad (\forall x \in X).$$

Proof. The proof makes repeated use of the gluing principle for Alberti representations, Theorem 2.32. Let $\{U_\alpha, Z_\alpha, N_\alpha, \iota_\alpha\}$ be as in the definition of a normal derivation 6.26. By taking an $L^\infty(\mu \llcorner U_\alpha)$ -partition of unity of each U_α , we can assume that the U_α are compact. By the gluing principle for Alberti representations (Theorem 2.32), it suffices to show that the desired representation exists for each $\mu \llcorner (U_\alpha \setminus N_{DF})$. In the following we can thus write X for U_α and drop the index α from the notation. Note also that the vector field $DF \circ \iota^{-1}$ can be extended to a vector field $v : Z \rightarrow \mathbb{R}^\infty$. By Theorem 2.14 in [Sch13] one can also show that the desired representation exists for $\iota_\#(\mu \llcorner (X \setminus N_{DF}))$; note that in this case the direction is determined by the function $F \circ \iota^{-1} : \iota(X) \rightarrow l^2$. In the following we will then identify $\iota(X)$ with X , $\iota_\# \mu$ with μ , and $\iota_\# D$ with D . We now take a MacShane extension

$$(6.34) \quad \tilde{F}_i : Z \rightarrow \mathbb{R}$$

of F_i with the same Lipschitz constant $\mathbf{L}(F_i)$ and then choose $c_i \in (0, 1)$ such that

$$(6.35) \quad \sum_i c_i^2 \mathbf{L}(F_i)^2 \leq 1.$$

In particular, the map $G : Z \rightarrow l^2$ with components $G_i = c_i \tilde{F}_i$ is 1-Lipschitz. Recalling the discussion before Definition 6.23, we also have, after choosing appropriate Borel representatives, that the components of the vector field DG satisfy:

$$(6.36) \quad DG_i = c_i DF_i.$$

Consider a fragment $\gamma : K \rightarrow X$. As l^2 has the Radon-Nikodym property, $F \circ \gamma$ and $G \circ \gamma$ are differentiable for $t \in Q \subset K$, where the Borel set Q satisfies $\mathcal{L}^1(K \setminus Q) = 0$. Moreover, at each point $t \in Q$ we have that $(F \circ \gamma)'(t)$ and $(G \circ \gamma)'(t)$ are determined by the derivatives $(F_i \circ \gamma)'(t)$ and $(G_i \circ \gamma)'(t)$ which are related by

$$(6.37) \quad (F_i \circ \gamma)'(t) = c_i (G_i \circ \gamma)'(t).$$

In particular, for $\lambda > 0$ the following equations are equivalent:

$$(6.38) \quad (F \circ \gamma)'(t) = \lambda DF(\gamma(t))$$

$$(6.39) \quad (G \circ \gamma)'(t) = \lambda DG(\gamma(t)),$$

and so we can replace F with G . Finally, we take another $L^\infty(\mu)$ -partition of unity to ensure that (4) holds. \square

The second ingredient in the proof of Theorem 6.31 is the following notion of strict convexity for the local norm in $\mathcal{X}(\mu)$.

Definition 6.40. The local norm $|\cdot|_{\mathcal{X}(\mu), \text{loc}}$ on $\mathcal{X}(\mu)$ is called **strictly convex** if the following holds: whenever one has that for derivations $D_1, D_2 \in \mathcal{X}(\mu)$ and for a Borel set U :

$$(6.41) \quad |D_1 + D_2|_{\mathcal{X}(\mu), \text{loc}}(x) = |D_1|_{\mathcal{X}(\mu), \text{loc}}(x) + |D_2|_{\mathcal{X}(\mu), \text{loc}}(x) \quad (\text{for } \mu\text{-a.e. } x \in U),$$

then $\chi_U D_1$ and $\chi_U D_2$, regarded as elements of $\mathcal{X}(\mu \llcorner U)$, are linearly dependent.

In Subsection 7.3 we show (Theorem 7.101) that it is always possible to perturb the metric on X in a biLipschitz way and obtain a strictly convex local norm on $\mathcal{X}(\mu)$. Therefore, for $\varepsilon > 0$, we can assume that the metric d on Z has been replaced by a metric $d^{(\varepsilon)}$ such that:

$$(6.42) \quad d \leq d^{(\varepsilon)} \leq (1 + \varepsilon)d,$$

and $|\cdot|_{\mathcal{X}(\|N\|), \text{loc}}^{(\varepsilon)}$ is strictly convex. We now apply Theorem 6.12 to obtain decompositions of N as in (6.13) and (6.14). We also construct countably many vector fields $w_j : Z \rightarrow \mathbb{R}^\infty$ such that:

- (1) For each j , there is $M_j \in \mathbb{N}$ such that $i > M_j$ implies $(w_j)_i = 0$, where $(w_j)_i$ is the i -th component of w_j .
- (2) If $DF(z) \neq 0$ and $\xi \in \mathbb{R}^\infty \setminus \{0\}$ is not a positive multiple of $DF(z)$, then $\langle w_j(z), \xi \rangle > 0$ for some j .
- (3) For each $z \in Z$, one has $\langle w_j(z), DF(z) \rangle \leq 0$.

We will denote by $w_0 : Z \rightarrow \mathbb{R}^\infty$ the null vector field.

We now introduce the set Ω_{fail} of those curves which, roughly speaking, meet X in a set of positive measure where the direction of $F \circ \gamma$ fails to be a positive multiple of DF . Specifically, we say that a curve $\gamma \in \text{Curves}(Z)$ belongs to Ω_{fail} if and only if there is a piece $\tilde{\gamma}$ of γ such that:

- (1) $F \circ \gamma$ is differentiable at each point $t \in \text{dom } \tilde{\gamma}$.
- (2) At each point $t \in \text{dom } \tilde{\gamma}$, the vector $(F \circ \gamma)'(t)$ is either 0 or, if it is nonzero, it is not a positive multiple of $DF \circ \gamma(t)$.
- (3) The piece $\tilde{\gamma}$ meets $X \setminus N_{DF}$ in positive mass measure: $\|[\tilde{\gamma}]\|(X \setminus N_{DF}) > 0$.

In general, the set Ω_{fail} is not Borel, but, after completing η , we will show that it becomes η -measurable. The goal is then to show that $\eta(\Omega_{\text{fail}}) = 0$. Note that the set Ω_{fail} is a countable union of the sets

$$(6.43) \quad \Omega_n(w_j) \subset \text{Curves}_n(Z)$$

defined as follows: $\gamma \in \text{Curves}_n(Z)$ belongs to $\Omega_n(w_j)$ if and only if there is a piece $\tilde{\gamma}$ of γ such that:

F1: $F \circ \gamma$ is differentiable at each point $t \in \text{dom } \tilde{\gamma}$.

F2: At each point $t \in \text{dom } \tilde{\gamma}$, if $j \neq 0$ one has $\langle (F \circ \gamma)'(t), w_j(\gamma(t)) \rangle \geq \frac{1}{n}$,
and if $j = 0$ one has $(F \circ \gamma)'(t) = 0$.

F3: The piece $\tilde{\gamma}$ meets $X \setminus N_{DF}$ in mass measure at least $1/n$: $\|[\tilde{\gamma}]\|(X \setminus N_{DF}) \geq \frac{1}{n}$.

We will thus study the measurability properties of each set $\Omega_n(w_j)$, which is the projection of

$$(6.44) \quad \Omega_n^{(1)}(w_j) = \left\{ (\gamma, \tilde{\gamma}) \in \text{Curves}_n(Z) \times \text{Pieces}_n(Z) : \begin{array}{l} \tilde{\gamma} \text{ is a piece of } \gamma \\ \text{and (F1), (F2) and (F3) hold} \end{array} \right\}$$

on $\text{Curves}_n(Z)$.

Lemma 6.45. *The set $\Omega_n^{(1)}(w_j)$ is of class Π_1^1 , i.e. coanalytic. Thus $\Omega_n(w_j)$ is of class Σ_2^1 and, moreover, there is a uniformizing function $\sigma_{j,n} : \Omega_n(w_j) \rightarrow \Omega_n^{(1)}(w_j)$ which is universally measurable and whose graph is of class Π_1^1 .*

Proof. We prove the Lemma for $j \neq 0$ as the case $j = 0$ requires a minor modification of the argument. Consider the set $\Omega_n^{(2)}(w_j) \subset \text{Curves}_n(Z) \times \text{Pieces}_n(Z) \times [0, 1]$ consisting of the triples $(\gamma, \tilde{\gamma}, t)$ such that:

- G1:** $\tilde{\gamma}$ is a piece of γ .
- G2:** $\|[\tilde{\gamma}]\|(X \setminus N_{DF}) \geq \frac{1}{n}$.
- G3:** either $t \notin \text{dom } \tilde{\gamma}$ or $t \in \text{dom } \tilde{\gamma}$ and $F \circ \gamma$ is differentiable at t with $\langle (F \circ \gamma)'(t), w_j(\gamma(t)) \rangle \geq \frac{1}{n}$.

We show that $\Omega_n^{(2)}(w_j)$ is Borel. First note that the set of couples $(\gamma, \tilde{\gamma})$ such that $\tilde{\gamma}$ is a piece of γ is closed in $\text{Curves}_n(Z) \times \text{Pieces}_n(Z)$. Second, as the map $\tilde{\gamma} \mapsto \|[\tilde{\gamma}]\|(X \setminus N_{DF})$ is Borel (6.21), the set of pieces with $\|[\tilde{\gamma}]\|(X \setminus N_{DF}) \geq \frac{1}{n}$ is Borel. Third, the set of pairs $(\tilde{\gamma}, t)$ with $t \in \text{dom } \tilde{\gamma}$ is closed. Therefore, we have only to show that the set

$$(6.46) \quad \tilde{\Omega} = \left\{ (\gamma, t) \in \text{Curves}_n(Z) \times [0, 1] : (F \circ \gamma)'(t) \text{ exists and } \langle (F \circ \gamma)'(t), w_j(\gamma(t)) \rangle \geq \frac{1}{n} \right\}$$

is Borel. Let \mathcal{S} denote a countable dense set of l^2 . We then have:

$$(6.47) \quad \tilde{\Omega} = \bigcap_{\varepsilon \in \mathbb{Q} \cap (0,1)} \bigcup_{\delta \in \mathbb{Q} \cap (0,1)} \bigcap_{s_1, s_2 \in \mathbb{Q} \cap (0,1)} \bigcup_{\xi \in \mathcal{S}} \left(\text{Curves}_n(Z) \times \left\{ t \in (0,1) : |t - s_1| \geq \delta \right. \right. \\ \left. \left. \text{or } |t - s_2| \geq \delta \right\} \cup S(\varepsilon, \delta, s_1, s_2, \xi) \right),$$

where $(\gamma, t) \in S(\varepsilon, \delta, s_1, s_2, \xi)$ if and only if the following inequalities hold:

$$(6.48) \quad |t - s_i| < \delta \quad (i = 1, 2)$$

$$(6.49) \quad \|F \circ \gamma(t)(s_1 - s_2) - (t - s_2)F \circ \gamma(s_1) + (t - s_1)F \circ \gamma(s_2)\|_{l^2} \leq \varepsilon |t - s_1| |t - s_2|$$

$$(6.50) \quad \|F \circ \gamma(t) - F \circ \gamma(s_1) - \xi(t - s_1)\|_{l^2} \leq \varepsilon |t - s_1|$$

$$(6.51) \quad \langle \xi, w_j(\gamma(t)) \rangle \geq \frac{1}{n} - \varepsilon.$$

We conclude that $S(\varepsilon, \delta, s_1, s_2, \xi)$ is Borel and so $\tilde{\Omega}$ is Borel, which completes the proof that $\Omega_n^{(2)}(w_j)$ is Borel. Note that $\Omega_n^{(1)}(w_j)$ is the coprojection of $\Omega_n^{(2)}(w_j)$ on $\text{Curves}_n(Z) \times \text{Pieces}_n(Z)$, which implies that $\Omega_n^{(1)}(w_j)$ is coanalytic. By the definition of the class Σ_2^1 , as $\Omega_n(w_j)$ is the projection of a conalytic set, it is of class Σ_2^1 . By the Σ_1^1 -determinacy [Kec95, Cor. 36.21], $\Omega_n(w_j)$ is universally measurable and there is a uniformizing function $\sigma_{j,n}$ as in the statement of this Lemma. \square

We now define maps

$$(6.52) \quad \Xi_{j,n} : \text{Curves}_n(Z) \rightarrow M_1(Z)$$

$$\gamma \mapsto \begin{cases} [\sigma_{j,n}(\gamma)] & \text{if } \gamma \in \Omega_n(w_j) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(6.53) \quad \Xi_{j,n}^c : \text{Curves}_n(Z) \rightarrow M_1(Z)$$

$$\gamma \mapsto \begin{cases} [\gamma] - [\sigma_{j,n}(\gamma)] & \text{if } \gamma \in \Omega_n(w_j) \\ [\gamma] & \text{otherwise.} \end{cases}$$

Note that for each $(f, \pi) \in \mathcal{B}^\infty(Z) \times \text{Lip}(Z)$ and each Borel set $E \subset Z$, the maps:

$$(6.54) \quad \gamma \mapsto \Xi_{j,n}(\gamma)(f d\pi)$$

$$(6.55) \quad \gamma \mapsto \Xi_{j,n}^c(\gamma)(f d\pi)$$

$$(6.56) \quad \gamma \mapsto \|\Xi_{j,n}(\gamma)\|(E)$$

$$(6.57) \quad \gamma \mapsto \|\Xi_{j,n}^c(\gamma)\|(E)$$

are universally measurable. In particular, they are η -measurable, as we assume that η is complete. Moreover, by definition of the maps $\Xi_{j,n}$ and $\Xi_{j,n}^c$, we have the relation:

$$(6.58) \quad [\gamma] = \Xi_{j,n}(\gamma) + \Xi_{j,n}^c(\gamma);$$

this implies that

$$(6.59) \quad \|[\gamma]\| \leq \|\Xi_{j,n}(\gamma)\| + \|\Xi_{j,n}^c(\gamma)\|;$$

however, for η -a.e. γ , if $\gamma \in \Omega_n(w_j)$, (6.15) implies that:

$$(6.60) \quad \begin{aligned} \|[\gamma]\|(Z) &= l(\gamma) = \int_0^1 \text{md } \gamma(t) dt \\ &= \int_{\text{dom } \sigma_{j,n}(\gamma)} \text{md } \gamma(t) dt + \int_{[0,1] \setminus \text{dom } \sigma_{j,n}(\gamma)} \text{md } \gamma(t) dt \\ &\geq \|\Xi_{j,n}(\gamma)\|(Z) + \|\Xi_{j,n}^c(\gamma)\|(Z), \end{aligned}$$

and thus, for η -a.e. γ , we have:

$$(6.61) \quad \|\gamma\| = \|\Xi_{j,n}(\gamma)\| + \|\Xi_{j,n}^c(\gamma)\|.$$

Lemma 6.62. *For each n and j we have that $\eta(\Omega_n(w_j)) = 0$.*

Proof of Lemma 6.62. We argue by contradiction assuming that $\eta(\Omega_n(w_j)) > 0$. Note that:

$$(6.63) \quad N = \underbrace{\int_{\text{Curves}(Z)} \Xi_{j,n}(\gamma) d\eta(\gamma)}_{T_{j,n}} + \underbrace{\int_{\text{Curves}(Z)} \Xi_{j,n}^c(\gamma) d\eta(\gamma)}_{T_{j,n}^c},$$

and, using (6.61),

$$(6.64) \quad \begin{aligned} \|N\|(Z) &= \int_{\text{Curves}(Z)} \|\gamma\|(Z) d\eta(\gamma) = \int_{\text{Curves}(Z)} \|\Xi_{j,n}(\gamma)\|(Z) d\eta(\gamma) \\ &\quad + \int_{\text{Curves}(Z)} \|\Xi_{j,n}^c(\gamma)\|(Z) d\eta(\gamma) \\ &\geq \|T_{j,n}\|(Z) + \|T_{j,n}^c\|(Z), \end{aligned}$$

where we used:

$$(6.65) \quad \int_{\text{Curves}(Z)} \|\Xi_{j,n}(\gamma)\|(Z) d\eta(\gamma) \geq \|T_{j,n}\|(Z),$$

$$(6.66) \quad \int_{\text{Curves}(Z)} \|\Xi_{j,n}^c(\gamma)\|(Z) d\eta(\gamma) \geq \|T_{j,n}^c\|(Z).$$

In particular, $T_{j,n}$ and $T_{j,n}^c$ are *complementary subcurrents* of N because (6.64) implies that

$$(6.67) \quad \|N\| = \|T_{j,n}\| + \|T_{j,n}^c\|.$$

Moreover, we also have that:

$$(6.68) \quad \|T_{j,n}\| = \int_{\text{Curves}(Z)} \|\Xi_{j,n}(\gamma)\| d\eta(\gamma),$$

$$(6.69) \quad \|T_{j,n}^c\| = \int_{\text{Curves}(Z)} \|\Xi_{j,n}^c(\gamma)\| d\eta(\gamma).$$

By Theorem 3.7 we find derivations $D_{j,n}, D_{j,n}^c \in \mathcal{X}(\|N\|)$ such that

$$(6.70) \quad T_{j,n}(fd\pi) = \int_Z f D_{j,n} \pi d\|N\|$$

$$(6.71) \quad T_{j,n}^c(fd\pi) = \int_Z f D_{j,n}^c \pi d\|N\|$$

$$(6.72) \quad \|T_{j,n}\| = |D_{j,n}|_{\mathcal{X}(\|N\|), \text{loc}}^{(\varepsilon)} \|N\|$$

$$(6.73) \quad \|T_{j,n}^c\| = |D_{j,n}^c|_{\mathcal{X}(\|N\|), \text{loc}}^{(\varepsilon)} \|N\|.$$

Note that (6.33) implies that the measures $\|N\| \llcorner X$ and μ are in the same measure class and we can thus identify the rings $L^\infty(\|N\| \llcorner X)$ and $L^\infty(\mu)$ and the modules $\mathcal{X}(\|N\| \llcorner X)$ and $\mathcal{X}(\mu)$. Having picked a Borel representative of $|D_{j,n}|_{\mathcal{X}(\|N\|), \text{loc}}^{(\varepsilon)}$ and letting

$$(6.74) \quad X_{j,n} = \left\{ x \in X \setminus N_{DF} : |D_{j,n}|_{\mathcal{X}(\|N\|), \text{loc}}^{(\varepsilon)}(x) > 0 \right\},$$

we show that $\mu(X_{j,n}) > 0$ by showing that $\|T_{j,n}\|(X \setminus N_{DF}) > 0$:

$$(6.75) \quad \|T_{j,n}\|(X \setminus N_{DF}) = \int_{\text{Curves}(Z)} \|\Xi_{j,n}(\gamma)\|(X \setminus N_{DF}) d\eta(\gamma) \geq \frac{1}{n} \eta(\Omega_n(w_j)) > 0.$$

We now combine (6.67), (6.72) and (6.73) with the strict convexity of $|\cdot|_{\mathcal{X}(\|N\|), \text{loc}}^{(\varepsilon)}$ and the fact that $|D_{j,n}|_{\mathcal{X}(\|N\|), \text{loc}}^{(\varepsilon)} > 0$ on $X_{j,n}$, to conclude that there is a nonnegative $\lambda_{j,n} \in \mathcal{B}^\infty(Z)$, which vanishes on $Z \setminus X_{j,n}$ and is such that:

$$(6.76) \quad \chi_{X_{j,n}} D_{j,n}^c = \lambda_{j,n} D_{j,n}.$$

We then conclude that

$$(6.77) \quad \chi_{X_{j,n}} D_N = (\chi_{X_{j,n}} + \lambda_{j,n}) D_{j,n}.$$

If $j = 0$ we have $\chi_{X_{0,n}} D_N F = 0$ which contradicts the fact that $\chi_{X_{0,n}} D F \neq 0$. For $j \neq 0$ we argue as follows: let M_j be the maximal index such that $(w_j)_{M_j} \neq 0$; we consider the 1-form $\omega = \sum_{k=1}^{M_j} (w_j)_k dF_k$ and let g denote a nonnegative continuous function; we have:

$$(6.78) \quad \int_Z g \langle w_j, D_{j,n} F \rangle d\|N\| = T_{j,n}(g\omega) = \int_{\text{Curves}(Z)} \Xi_{j,n}(\gamma)(g\omega) d\eta(\gamma);$$

now, if $\gamma \in \Omega_n(w_j)$, $\sum_{k=1}^{M_j} (w_j)_k (\gamma(t)) (F_k \circ \gamma)'(t) \geq 1/n$ for $t \in \text{dom } \sigma_{j,n}$, which implies:

$$(6.79) \quad \int_Z g \langle w_j, D_{j,n} F \rangle d\|N\| \geq \frac{1}{n} \int_{\Omega_n(w_j)} d\eta(\gamma) \int_{\text{dom } \sigma_{j,n}} g \circ \gamma(t) dt;$$

as the curves in $\Omega_n(w_j)$ are n -Lipschitz and because of (6.22), we obtain

$$(6.80) \quad \begin{aligned} \int_Z g \langle w_j, D_{j,n} F \rangle d\|N\| &\geq \frac{1}{n^2} \int_{\Omega_n(w_j)} d\eta(\gamma) \int_{\text{dom } \sigma_{j,n}} g \circ \gamma(t) \text{md } \gamma(t) dt \\ &\geq \frac{1}{n^2} \int_{\Omega_n(w_j)} d\eta(\gamma) \int_Z g d\|\Xi_{j,n}(\gamma)\| \\ &= \frac{1}{n^2} \int_Z g d\|T_{j,n}\| \\ &= \frac{1}{n^2} \int_Z g |D_{j,n}|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)} d\|N\|. \end{aligned}$$

From (6.80) we conclude that $\langle w_j, D_{j,n} F \rangle > 0$ on $X_{j,n}$; moreover, from (6.77) we obtain $\langle w_j, D F \rangle > 0$ on $X_{j,n}$, but this contradicts the choice of w_j . Thus, $\eta(\Omega_n(w_j)) = 0$. \square

Proof of Theorem 6.31. By Lemma 6.62 we have $\eta(\Omega_n(w_j)) = 0$ which implies $\eta(\Omega_{\text{fail}}) = 0$. Therefore, for η -a.e. γ and \mathcal{L}^1 -a.e. t , $(F \circ \gamma)'(t)$ is a positive multiple of $DF(\gamma(t))$. The desired Alberti representation is then obtained using the measure η . Specifically, let

$$(6.81) \quad \text{Rep} : \text{Curves}(Z) \rightarrow \text{Frag}(Z)$$

be a Borel map which reparametrizes each $\gamma \in \text{Curves}(Z)$ to a 1-Lipschitz map $\text{Rep} : [0, \lceil \mathbf{L}(\gamma) \rceil] \rightarrow Z$. Note that up to passing to a Borel $L^\infty(\mu)$ -partition of unity

we can assume that the set $X \setminus N_{DF}$ is compact; we now consider the measure:

$$(6.82) \quad \nu_1 = \int_{\text{Curves}(Z)} \|[\text{Rep}(\gamma)]\| d\eta(\gamma) = \int_{\text{Frag}(Z)} \|[\gamma]\| d(\text{Rep}_\# \eta)(\gamma)$$

and observe that $\|N\| \ll \nu_1$ and that $\text{Rep}_\# \eta$ is concentrated on the set of 1-Lipschitz fragments. We now let

$$(6.83) \quad \text{Frag}(Z, X \setminus N_{DF}) = \{\gamma \in \text{Frag}(Z) : \gamma^{-1}(X \setminus N_{DF}) \neq \emptyset\}$$

and note that $\text{Frag}(Z, X \setminus N_{DF})$ is a closed subset of $\text{Frag}(Z)$. An argument similar to that of Lemma 2.21 in [Sch13] shows that the map:

$$(6.84) \quad \begin{aligned} \text{Red}_{X \setminus N_{DF}} : \text{Frag}(Z, X \setminus N_{DF}) &\rightarrow \text{Frag}(X) \\ \gamma &\mapsto \gamma|_{\gamma^{-1}(X \setminus N_{DF})} \end{aligned}$$

is Borel. We now consider the measure

$$(6.85) \quad \nu_2 = \int_{\text{Frag}(Z, X \setminus N_{DF})} \|[\text{Red}_{X \setminus N_{DF}}(\gamma)]\| d(\text{Rep}_\# \eta)(\gamma) = \int_{\text{Frag}(X)} \|[\gamma]\| \underbrace{d(\text{Red}_{X \setminus N_{DF}})_\# \text{Rep}_\# \eta(\gamma)}_{\eta_2}$$

and note that $\mu \ll \nu_2$; an Alberti representation as in the statement of this Theorem is then:

$$(6.86) \quad \mu = \int_{\text{Frag}(X)} (\text{Rep}_\# \eta)(\text{Frag}(Z, X \setminus N_{DF})) \|[\gamma] \frac{d\mu}{d\nu_2}\| \frac{d\eta_2(\gamma)}{(\text{Rep}_\# \eta)(\text{Frag}(Z, X \setminus N_{DF}))}.$$

□

7. TECHNICAL TOOLS

7.1. Exterior Products. In this Subsection we define the exterior powers in different categories:

- In the category **Ban**, whose objects are Banach spaces and whose morphisms are bounded linear maps;
- In the category ${}^\infty_\mu \mathbf{Mod}$, whose objects are $L^\infty(\mu)$ -modules and whose morphisms are bounded module homomorphisms;
- In the category ${}^\infty_\mu \mathbf{Mod}_{\text{loc}}$, whose objects are $L^\infty(\mu)$ -normed modules and whose morphisms are bounded module homomorphisms.

In the following, if Z is a Banach space, we will denote by Z^* its dual. If Z is also an $L^\infty(\mu)$ -module, we will denote by Z' the dual module; note that Z^* and Z' are, in general, different (Example 7.13).

Definition 7.1. For Banach spaces Z and W , let $\text{Alt}_k(Z; W)$ denote the set of alternating multilinear maps $\varphi : Z^k \rightarrow W$ which are bounded with respect to the norm:

$$(7.2) \quad \|\varphi\|_{\text{Alt}_k(Z; W)} = \sup \left\{ \|\varphi(m_1, \dots, m_k)\|_W : \max_{i=1, \dots, k} \|m_i\|_Z \leq 1 \right\}.$$

Definition 7.3. For $L^\infty(\mu)$ -modules M and N , let $\text{Alt}_k(M; N)$ denote the set of alternating $L^\infty(\mu)$ -multilinear maps $\varphi : M^k \rightarrow N$ which are bounded with respect to the norm:

$$(7.4) \quad \|\varphi\|_{\text{Alt}_k(M; N)} = \sup \left\{ \|\varphi(m_1, \dots, m_k)\|_N : \max_{i=1, \dots, k} \|m_i\|_M \leq 1 \right\}.$$

Definition 7.5. Let Z be an Banach space. The **projective k -th power of Z in the category \mathbf{Ban}** is a pair $(\text{Ext}^k Z, \pi)$, where $\text{Ext}^k Z$ is an Banach space and $\pi \in \text{Alt}_k(Z; \text{Ext}^k Z)$, which satisfies the following universal property: for each $\varphi \in \text{Alt}_k(Z; W)$, where W is an Banach space, there is a unique $\hat{\varphi} \in \text{hom}(\text{Ext}^k Z, W)$ which makes the following diagram commutative:

$$(7.6) \quad \begin{array}{ccc} Z^k & \xrightarrow{\pi} & \text{Ext}^k Z \\ \varphi \downarrow & \swarrow \hat{\varphi} & \\ W & & \end{array}$$

and such that $\|\hat{\varphi}\|_{\text{hom}(\text{Ext}^k Z, W)} = \|\varphi\|_{\text{Alt}_k(Z; W)}$.

Definition 7.7. Let M be an $L^\infty(\mu)$ -module. The **projective k -th power of M in the category ${}^\infty\mathbf{Mod}$** is a pair $(\text{Ext}_\mu^k M, \pi)$, where $\text{Ext}_\mu^k M$ is an $L^\infty(\mu)$ -module and $\pi \in \text{Alt}_k(M; \text{Ext}_\mu^k M)$, which satisfies the following universal property: for each $\varphi \in \text{Alt}_k(M; N)$, where N is an $L^\infty(\mu)$ -module, there is a unique $\hat{\varphi} \in \text{hom}(\text{Ext}_\mu^k M, N)$ which makes the following diagram commutative:

$$(7.8) \quad \begin{array}{ccc} M^k & \xrightarrow{\pi} & \text{Ext}_\mu^k M \\ \varphi \downarrow & \swarrow \hat{\varphi} & \\ N & & \end{array}$$

and such that $\|\hat{\varphi}\|_{\text{hom}(\text{Ext}_\mu^k M, N)} = \|\varphi\|_{\text{Alt}_k(M; N)}$.

Definition 7.9. Let M be an $L^\infty(\mu)$ -normed module. The **projective k -th power of M in the category ${}^\infty\mathbf{Mod}_{\text{loc}}$** is a pair $(\text{Ext}_{\mu, \text{loc}}^k M, \pi)$, where $\text{Ext}_{\mu, \text{loc}}^k M$ is an $L^\infty(\mu)$ -normed module and $\pi \in \text{Alt}_k(M; \text{Ext}_{\mu, \text{loc}}^k M)$, which satisfies the following universal property: for each $\varphi \in \text{Alt}_k(M; N)$, where N is an $L^\infty(\mu)$ -normed module, there is a unique $\hat{\varphi} \in \text{hom}(\text{Ext}_{\mu, \text{loc}}^k M, N)$ which makes the following diagram commutative:

$$(7.10) \quad \begin{array}{ccc} M^k & \xrightarrow{\pi} & \text{Ext}_{\mu, \text{loc}}^k M \\ \varphi \downarrow & \swarrow \hat{\varphi} & \\ N & & \end{array}$$

and such that $\|\hat{\varphi}\|_{\text{hom}(\text{Ext}_{\mu, \text{loc}}^k M, N)} = \|\varphi\|_{\text{Alt}_k(M; N)}$.

We now present some illustrative examples. Recall that an *atom* for a measure μ is a positive measure set A such that for each proper subset B , $\mu(B) = 0$; note that if A is an atom for a Radon measure μ , A is a singleton. A measure without atoms is called *non-atomic*; in particular, a Radon measure μ is non-atomic if and only if $\mu(\{x\}) = 0$ for each singleton $\{x\}$. We now recall the Sierpiński's Theorem [Fry04, pg. 39]:

Theorem 7.11. *If μ is a non-atomic measure on a space X with $\mu(X) = c < \infty$ and Σ is the σ -algebra of μ -measurable sets, then there is a function $S : [0, c] \rightarrow \Sigma$ which is monotone with respect to inclusion and is a right inverse of $\mu : \Sigma \rightarrow [0, c]$.*

In the following we will assume $p \in [1, \infty)$.

Example 7.12. If μ is a finite sum of Dirac masses, $L^p(\mu)$ can be identified with $L^\infty(\mu)$ and so is free of rank 1.

Suppose that μ is non-atomic; in particular, by Theorem 7.11, given any positive measure set U , it is possible to find $f \in L^p(\mu \llcorner U)$ with $\|f\|_{L^p(\mu)} \leq 1$ and $\forall n$ $\mu(x \in U : |f(x)| > n) > 0$. Suppose that $L^p(\mu)$ was generated by f_1, \dots, f_N ; then there would be a set of positive measure U on which the f_i , and hence all the element in $L^p(\mu)$ would be uniformly bounded, leading to a contradiction.

However, any two elements of $L^p(\mu)$ are linearly dependent over $L^\infty(\mu)$. If $f \in L^p(\mu)$ vanishes on a set of positive measure U , it suffices to note that f is annihilated by χ_U . If f and g are nowhere vanishing, there is a positive measure set U on which $0 < c_0 < |f|, |g| < c_1 < \infty$; then it is possible to find $\lambda \in L^\infty(\mu)$ with $\chi_U f + \lambda g = 0$. In particular, if $f \in L^p(\mu)$ is nowhere vanishing, the algebraic submodule generated by f is dense.

Example 7.13. Given an $L^\infty(\mu)$ -module M , there are two notions of dual. The dual module of M , $\text{hom}(M, L^\infty(\mu)) = M'$ is an $L^\infty(\mu)$ -normed module. However, the dual Banach space of M , M^* , is also an $L^\infty(\mu)$ -module if we let

$$(7.14) \quad \lambda \cdot \varphi(m) = \varphi(\lambda m).$$

For example, if $M = L^p(\mu)$, then $M^* = L^q(\mu)$.

We show that if μ is non-atomic, then the algebraic dual of M (and hence M') is trivial. By replacing μ by $\mu \llcorner U$, where U is a set of positive measure, we can assume that μ is finite, so that $L^\infty(\mu) \subset L^p(\mu)$; let $\Phi : L^p(\mu) \rightarrow L^\infty(\mu)$ be a module homomorphism; supposing that $\Phi(1) \neq 0$, we can use Theorem 7.11 to find $f \in L^p(\mu)$ and μ -measurable sets U_n such that:

- for each n , $\Phi(1)\chi_{U_n} f \in L^\infty(\mu)$;
- for each n :

$$(7.15) \quad \mu(\{x \in U_n : |\Phi(1)\chi_{U_n} f|(x) > n\}) > 0.$$

Note that

$$(7.16) \quad \chi_{U_n} \Phi(f) = \Phi(\chi_{U_n} f) = \Phi(1)\chi_{U_n} f$$

shows that $\Phi(f) \notin L^\infty(\mu)$, a contradiction. Thus $\Phi(1) = 0$ implying $\Phi = 0$. In this case, the dual module of $L^p(\mu)$ is trivial.

Suppose now that μ is a countable sum of Dirac masses: $\mu = \sum_n c_n \delta_{p_n}$, so that a function f is in the unit ball of $L^p(\mu)$ iff

$$(7.17) \quad \sum_n |f_n|^p c_n \leq 1 \quad (f_n = f(p_n));$$

as $\varphi \in M'$ is determined by the values on the functions δ_{p_n} , we can identify it with the module of those sequences $\{\lambda_n\}$ for which there is a $C_\lambda > 0$ with $|\lambda_n| \leq C_\lambda (c_n)^{1/p}$; the norm is $\sup_n |\lambda_n| / (c_n)^{1/p}$.

Example 7.18. For an $L^\infty(\mu)$ -module N , $\text{Alt}_k(L^p(\mu); N)$ is trivial and the case $k = 1$ was treated in Example 7.13. Note that $\Omega = L^p(\mu) \cap L^\infty(\mu)$ is a dense algebraic submodule of $L^p(\mu)$; in particular, $T \in \text{Alt}_k(L^p(\mu); N)$ is determined by

its values on Ω^k ; however, Ω is also an algebraic submodule of $L^\infty(\mu)$; in particular, there is an alternating multilinear mapping $\tilde{T} = (L^\infty(\mu))^k \rightarrow N$ which agrees with T on Ω^k and vanishes elsewhere; for $k > 1$, $\tilde{T} = 0$ and so $T = 0$.

Note that the nullity of $\text{Alt}_k(L^p(\mu); N)$ for N an $L^\infty(\mu)$ -normed module, implies that $\text{Ext}_{\mu, \text{loc}}^k L^p(\mu) = 0$.

Example 7.19. Let $\|\cdot\|$ a norm on \mathbb{R}^n ; on $\bigwedge^k \mathbb{R}^n$ we consider the norm:

$$(7.20) \quad \|\omega\| = \inf \left\{ \sum_{i \in I} \|v_{i_1}\| \cdots \|v_{i_k}\| : \omega = \sum_{i \in I} v_{i_1} \wedge \cdots \wedge v_{i_k} \right\}.$$

We will denote by μ a non-atomic Radon measure.

We claim that $\text{Ext}_{\mu, \text{loc}}^k L^p(\mu; \mathbb{R}^n)$ is trivial. By the Hahn-Banach Theorem, it suffices to show that $\text{Alt}_k(L^p(\mu; \mathbb{R}^n); L^\infty(\mu))$ is trivial; suppose that for U a Borel set of finite measure and $\{v_i\}_{i=1}^k \subset \mathbb{R}^n$ independent vectors we had

$$(7.21) \quad T(\chi_U v_1, \dots, \chi_U v_k) \neq 0$$

where $T \in \text{Alt}_k(L^p(\mu; \mathbb{R}^n); L^\infty(\mu))$; arguing as in Example 7.13, we would reach a contradiction.

However we show that $\text{Ext}_{\mu}^k L^p(\mu; \mathbb{R}^n)$ can be identified with $L^{p/k}(\mu; \bigwedge^k \mathbb{R}^n)$ under the assumption $p \in [k, \infty)$. By Hölder's inequality, the multilinear alternating map

$$(7.22) \quad \begin{aligned} E : (L^p(\mu; \mathbb{R}^n))^k &\rightarrow L^{p/k}(\mu; \bigwedge^k \mathbb{R}^n) \\ (f_1, \dots, f_k) &\mapsto f_1 \wedge \cdots \wedge f_k \end{aligned}$$

has norm at most 1. For $\psi \in L^{p/k}(\mu)$ define:

$$(7.23) \quad \begin{aligned} T_\psi : (\mathbb{R}^n)^k &\rightarrow N \\ (v_1, \dots, v_k) &\mapsto T(\text{sgn } \psi |\psi|^{1/k} v_1, |\psi|^{1/k} v_2, \dots, |\psi|^{1/k} v_k); \end{aligned}$$

the map T_ψ is multilinear and alternating (as a map of vector spaces); let $\hat{T}_\psi : \bigwedge^k \mathbb{R}^n \rightarrow N$ denote the corresponding linear map given by the universal property of $\bigwedge^k \mathbb{R}^n$. For $\omega \in \bigwedge^k \mathbb{R}^n$ note that

$$(7.24) \quad \|\hat{T}_\psi(\omega)\| \leq \|T\| \|\psi\|_{p/k} \|\omega\|;$$

the multilinearity of T can be used to show that $\hat{T}_{\psi_1 + \psi_2} = \hat{T}_{\psi_1} + \hat{T}_{\psi_2}$; the multilinearity of T over $L^\infty(\mu)$ also implies that $\hat{T}_{\lambda\psi} = \lambda \hat{T}_\psi$. Note that any $f \in L^{p/k}(\mu; \bigwedge^k \mathbb{R}^n)$ can be written as

$$(7.25) \quad f = \sum_i f_i \omega_i,$$

where $\{\omega_i\}$ is a basis of $\bigwedge^k \mathbb{R}^n$; in particular, we can define $\hat{T} : L^{p/k}(\mu; \bigwedge^k \mathbb{R}^n) \rightarrow N$ by

$$(7.26) \quad \hat{T}(f) = \sum_i \hat{T}_{f_i}(\omega_i);$$

note that the definition is well-posed because any f has a unique expression (7.25). Furthermore, the definition of \hat{T} does not depend on the choice of the basis $\{\omega_i\}$ as

can be verified by linear algebra. Note also that the map \hat{T} is linear over $L^\infty(\mu)$. Using (7.26) we conclude that there is a dimensional constant $C_{n,k}$ such that

$$(7.27) \quad \|\hat{T}\| \leq C_{n,k} \|T\|.$$

As simple functions are dense in $L^p(\mu; \mathbb{R}^n)$ and using (7.27) we conclude that

$$(7.28) \quad T(f_1, \dots, f_k) = \hat{T}(f_1 \wedge \dots \wedge f_k);$$

this implies that $\|\hat{T}\| = \|T\|$ and that $L^{p/k}(\mu; \bigwedge^k \mathbb{R}^n)$ is the exterior k -power of $L^p(\mu; \mathbb{R}^n)$.

In the remainder of this section we assume that μ is a Radon measure. The following Lemma summarizes some properties of the Banach space $\text{Alt}_k(M; N)$.

Lemma 7.29. *Let M, N be $L^\infty(\mu)$ -modules; then $\text{Alt}_k(M; N)$ is an $L^\infty(\mu)$ -module and it is an $L^\infty(\mu)$ -normed module if N is an $L^\infty(\mu)$ -normed module. Moreover if M and N are $L^\infty(\mu)$ -normed modules, for $\varphi \in \text{Alt}_k(M; N)$ and $\{m_i\}_{i=1}^k \subset M$*

$$(7.30) \quad |\varphi(m_1, \dots, m_k)|_{N, \text{loc}} \leq |\varphi|_{\text{Alt}_k(M; N), \text{loc}} |m_1|_{M, \text{loc}} \cdots |m_k|_{M, \text{loc}}.$$

Proof of Lemma 7.29. The fact that $\text{Alt}_k(M; N)$ is a Banach space with the norm $\|\cdot\|_{\text{Alt}_k(M; N)}$ follows from a standard argument. For $(\varphi, \lambda) \in \text{Alt}_k(M; N) \times L^\infty(\mu)$ the product $\lambda\varphi$ can be defined by:

$$(7.31) \quad \lambda\varphi(m_1, \dots, m_k) = \varphi(m_1, \dots, \lambda m_i, \dots, m_k) \quad (\text{any choice of } i)$$

which makes $\text{Alt}_k(M; N)$ an $L^\infty(\mu)$ -module.

If N is an $L^\infty(\mu)$ -normed module, for a μ -measurable subset $U \subset X$, we have

$$(7.32) \quad \begin{aligned} \|\varphi\|_{\text{Alt}_k(M; N)} &= \sup_{\|m_i\|_M \leq 1} \|\varphi(m_1, \dots, m_k)\|_N \\ &= \sup_{\|m_i\|_M \leq 1} \max(\|\chi_U \varphi(m_1, \dots, m_k)\|_N, \|\chi_{X \setminus U} \varphi(m_1, \dots, m_k)\|_N) \\ &= \max \left(\sup_{\|m_i\|_M \leq 1} \|(\chi_U \varphi)(m_1, \dots, m_k)\|, \sup_{\|m_i\|_M \leq 1} \|(\chi_{X \setminus U} \varphi)(m_1, \dots, m_k)\| \right) \\ &= \max(\|\chi_U \varphi\|_{\text{Alt}_k(M; N)}, \|\chi_{X \setminus U} \varphi\|_{\text{Alt}_k(M; N)}); \end{aligned}$$

by [Wea00, Thm. 2] $\text{Alt}_k(M; N)$ is an $L^\infty(\mu)$ -normed module.

We now show (7.30) under the assumption that M and N are $L^\infty(\mu)$ -normed modules. By [Wea00, Cor. 6] we can find $\Phi_{m_1, \dots, m_k} \in N'$ with $\|\Phi_{m_1, \dots, m_k}\|_{N'} \leq 1$ and

$$(7.33) \quad |\varphi(m_1, \dots, m_k)|_{N, \text{loc}} = \langle \Phi_{m_1, \dots, m_k}, \varphi(m_1, \dots, m_k) \rangle;$$

let $\xi \in \text{Alt}_k(M; L^\infty(\mu))$ be defined by

$$(7.34) \quad \xi(\tilde{m}_1, \dots, \tilde{m}_k) = \langle \Phi_{m_1, \dots, m_k}, \varphi(\tilde{m}_1, \dots, \tilde{m}_k) \rangle;$$

for $\varepsilon > 0$ we can find an $L^\infty(\mu)$ -partition of unity $\{U_\alpha\}$ such that for $x \in U_\alpha$ and $1 \leq i \leq k$,

$$(7.35) \quad |\xi|_{\text{Alt}_k(M; L^\infty(\mu)), \text{loc}}(x) \in (\|\chi_{U_\alpha} \xi\|_{\text{Alt}_k(M; L^\infty(\mu))} - \varepsilon, \|\chi_{U_\alpha} \xi\|_{\text{Alt}_k(M; L^\infty(\mu))})];$$

$$(7.36) \quad |m_i|_{M, \text{loc}}(x) \in (\|\chi_{U_\alpha} m_i\|_M - \varepsilon, \|\chi_{U_\alpha} m_i\|_M).$$

Using the definition of norm in $\text{Alt}_k(M; L^\infty(\mu \llcorner U_\alpha))$ and (7.35) and (7.36),

$$\begin{aligned}
 \xi(m_1, \dots, m_k) &= \sum_{\alpha} \chi_{U_\alpha} \xi(m_1, \dots, m_k) \\
 &= \sum_{\alpha} (\chi_{U_\alpha} \xi)(\chi_{U_\alpha} m_1, \dots, \chi_{U_\alpha} m_k) \\
 &\leq \sum_{\alpha} \chi_{U_\alpha} \|\chi_{U_\alpha} \xi\|_{\text{Alt}_k(M; L^\infty(\mu))} \|\chi_{U_\alpha} m_1\|_M \cdots \|\chi_{U_\alpha} m_k\|_M \\
 (7.37) \quad &\leq \sum_{\alpha} \chi_{U_\alpha} \left(|\xi|_{\text{Alt}_k(M; L^\infty(\mu)), \text{loc}} + \varepsilon \right) \prod_{i=1}^k \left(|m_i|_{M, \text{loc}} + \varepsilon \right) \\
 &= \left(|\xi|_{\text{Alt}_k(M; L^\infty(\mu)), \text{loc}} + \varepsilon \right) \prod_{i=1}^k \left(|m_i|_{M, \text{loc}} + \varepsilon \right).
 \end{aligned}$$

Note that (7.30) follows from (7.37) letting $\varepsilon \searrow 0$ provided we show

$$(7.38) \quad |\xi|_{\text{Alt}_k(M; L^\infty(\mu)), \text{loc}} \leq |\varphi|_{\text{Alt}_k(M; N), \text{loc}}.$$

As for each μ -measurable U we have

$$(7.39) \quad \|\chi_U \xi\|_{\text{Alt}_k(M; L^\infty(\mu))} \leq \|\chi_U \varphi\|_{\text{Alt}_k(M; N)},$$

(7.38) holds. \square

We now prove the existence of the exterior powers in the category **Ban**.

Theorem 7.40. *For Z a Banach space, the k -th exterior power in the category **Ban** exists and can be realized as a closed subspace of the dual space $\text{Alt}_k(M; \mathbb{R})^*$; moreover, the algebraic k -th exterior power $\bigwedge^k Z$ is dense in $\text{Ext}^k Z$.*

proof of Theorem 7.40. For $\varphi \in \text{Alt}_k(Z; \mathbb{R})$ let $\tilde{\varphi} : \bigwedge^k Z \rightarrow \mathbb{R}$ denote the unique linear map corresponding to φ given by the universal property of $\bigwedge^k Z$. In particular, we obtain a map E from $\bigwedge^k Z$ to the algebraic dual of $\text{Alt}_k(Z; \mathbb{R})$ by letting $\langle E(w), \varphi \rangle = \tilde{\varphi}(w)$. We now show that $E(w)$ is a bounded functional. Let

$$(7.41) \quad w = \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k}$$

and note that

$$\begin{aligned}
 \left\| \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k} \right\|_{(\text{Alt}_k(Z; \mathbb{R}))^*} &= \sup_{\|\varphi\|_{\text{Alt}_k(Z; \mathbb{R})} \leq 1} \left| \left\langle \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k}, \varphi \right\rangle \right| \\
 (7.42) \quad &\leq \sup_{\|\varphi\|_{\text{Alt}_k(Z; \mathbb{R})} \leq 1} \sum_{i \in I} |\varphi(z_{i_1}, \dots, z_{i_k})| \\
 &\leq \sum_{i \in I} \|z_{i_1}\|_X \cdots \|z_{i_k}\|_X.
 \end{aligned}$$

We now show that E is injective; suppose $w \neq 0$; let Z_0 denote the linear span of $\Omega = \{z_{i_j} : j = 1, \dots, k; i \in I\}$ so that Z_0 is a finite dimensional vector space of dimension $L \geq k$. Having chosen a basis $\{v_\alpha\}_{\alpha=1}^L$ of Z_0 , without loss of generality we can assume that

$$(7.43) \quad w = \sum_{j \in \Lambda_{k,L}} c_j v_{j_1} \wedge \cdots \wedge v_{j_k}$$

with $c_{(1,\dots,k)} \neq 0$. If $\{v_\alpha^*\}_{\alpha=1}^L$ is the dual basis of $\{v_\alpha\}_{\alpha=1}^L$, by the Hahn-Banach Theorem the functionals v_α^* can be extended to elements of Z^* ; in particular,

$$(7.44) \quad \begin{aligned} \Xi : Z^k &\rightarrow \mathbb{R} \\ (z_1, \dots, z_k) &\mapsto \det((\langle v_\alpha^*, z_i \rangle)_{\alpha,i=1}^k) \end{aligned}$$

defines an element of $\text{Alt}_k(Z; \mathbb{R})$ and

$$(7.45) \quad \langle E(w), \Xi \rangle = c_{(1,\dots,k)} \neq 0$$

showing that E is injective.

We can thus identify $\bigwedge^k Z$ with a linear subspace of $\text{Alt}_k(Z; \mathbb{R})^*$ and we will denote its completion in the $\|\cdot\|_{(\text{Alt}_k(Z; \mathbb{R}))^*}$ norm by $\text{Ext}^k Z$. The map π is defined by

$$(7.46) \quad \pi(z_1, \dots, z_k) = z_1 \wedge \dots \wedge z_k;$$

note that π is alternating and multilinear and (7.42) shows that it is bounded. Let $\varphi \in \text{Alt}_k(Z; W)$ and define $\hat{\varphi} : \bigwedge^k Z \rightarrow W$ by

$$(7.47) \quad \hat{\varphi} \left(\sum_{i \in I} z_{i_1} \wedge \dots \wedge z_{i_k} \right) = \sum_{i \in I} \varphi(z_{i_1}, \dots, z_{i_k});$$

this is well-defined because φ is alternating multilinear and because of the universal property of $\bigwedge^k Z$. In order to show that $\hat{\varphi}$ has a unique extension $\hat{\varphi} : \text{Ext}^k Z \rightarrow W$, it suffices to show that $\hat{\varphi}$ is bounded:

$$(7.48) \quad \begin{aligned} \left\| \hat{\varphi} \left(\sum_{i \in I} z_{i_1} \wedge \dots \wedge z_{i_k} \right) \right\|_W &= \sup_{w^* \in W^* : \|w^*\|_{W^*} \leq 1} \left\langle w^*, \hat{\varphi} \left(\sum_{i \in I} z_{i_1} \wedge \dots \wedge z_{i_k} \right) \right\rangle \\ &= \|\varphi\|_{\text{Alt}_k(Z; W)} \sup_{\substack{w^* \in W^* : \\ \|w^*\|_{W^*} \leq 1}} \sum_{i \in I} \left\langle w^*, \frac{1}{\|\varphi\|_{\text{Alt}_k(Z; W)}} \varphi(z_{i_1}, \dots, z_{i_k}) \right\rangle \\ &\leq \|\varphi\|_{\text{Alt}_k(Z; W)} \sup_{\substack{\tau \in \text{Alt}_k(Z; \mathbb{R}) : \\ \|\tau\|_{\text{Alt}_k(Z; \mathbb{R})} \leq 1}} \left\langle \tau, \sum_{i \in I} z_{i_1} \wedge \dots \wedge z_{i_k} \right\rangle \\ &\leq \|\varphi\|_{\text{Alt}_k(Z; W)} \left\| \sum_{i \in I} z_{i_1} \wedge \dots \wedge z_{i_k} \right\|_{(\text{Alt}_k(Z; \mathbb{R}))^*}. \end{aligned}$$

Note that (7.48) shows that

$$(7.49) \quad \|\hat{\varphi}\|_{\text{hom}(\text{Ext}^k Z, W)} \leq \|\varphi\|_{\text{Alt}_k(Z; W)};$$

for the reverse inequality, observe that for each $\varepsilon > 0$, there are $z_i \in Z$ ($i \in \{1, \dots, k\}$) such that $\|z_i\|_Z \leq 1$ and

$$(7.50) \quad \|\varphi\|_{\text{Alt}_k(Z; W)} < \varepsilon + \|\varphi(z_1, \dots, z_k)\|_Z;$$

but

$$(7.51) \quad \varphi(z_1, \dots, z_k) = \hat{\varphi}(z_1 \wedge \dots \wedge z_k)$$

and by (7.42)

$$(7.52) \quad \|z_1 \wedge \dots \wedge z_k\|_{\bigwedge^k Z} \leq 1;$$

thus

$$(7.53) \quad \|\varphi\|_{\text{Alt}_k(Z;W)} < \varepsilon + \|\hat{\varphi}\|_{\text{hom}(\text{Ext}^k Z, W)}.$$

□

We now turn to the existence of exterior powers in the category ${}^\infty_\mu \mathbf{Mod}_{\text{loc}}$.

Theorem 7.54. *For M an $L^\infty(\mu)$ -normed module, the k -th exterior power in the category ${}^\infty_\mu \mathbf{Mod}_{\text{loc}}$ exists and can be realized as a closed submodule of the dual module $\text{Alt}_k(M; L^\infty(\mu))'$; moreover, the algebraic k -th exterior power ${}_{L^\infty(\mu)} \bigwedge^k M$ is dense in $\text{Ext}^k_{\mu, \text{loc}} M$.*

Proof of Theorem 7.54. Part of the proof is similar to the Banach space case (Theorem 7.40). For $\varphi \in \text{Alt}_k(M; L^\infty(\mu))'$ let $\tilde{\varphi} : {}_{L^\infty(\mu)} \bigwedge^k M \rightarrow L^\infty(\mu)$ denote the unique module homomorphism corresponding to φ given by the universal property of ${}_{L^\infty(\mu)} \bigwedge^k M$. The same estimate (7.42) used in the Banach space case shows that the map:

$$(7.55) \quad E : {}_{L^\infty(\mu)} \bigwedge^k M \rightarrow \text{Alt}_k(M; L^\infty(\mu))'$$

sending $w \in {}_{L^\infty(\mu)} \bigwedge^k M$ to the functional $E(w)$ satisfying

$$(7.56) \quad \langle E(w), \varphi \rangle = \tilde{\varphi}(w),$$

is well-defined.

We now show that E is injective. Let

$$(7.57) \quad w = \sum_{i \in I} m_{i_1} \wedge \cdots \wedge m_{i_k} \neq 0$$

and M_0 the $L^\infty(\mu)$ -submodule of M generated by the finite set

$$(7.58) \quad \Omega = \{m_{i_j} : j = 1, \dots, k; i \in I\}.$$

By [Wea00, Lem. 9] there are disjoint measurable sets $\{U_i\}_{i=1}^{\#\Omega}$ such that

$$(7.59) \quad 1 = \sum_{i=1}^{\#\Omega} \chi_{U_i},$$

and if $\mu(U_i) > 0$, then $\chi_{U_i} M_0$, regarded as an $L^\infty(\mu \lfloor U_i)$ -module, is free of rank i ; as we are assuming $w \neq 0$, $\chi_{U_L} w \neq 0$ for some index $L \geq k$. Let $\{\tilde{m}_i\}_{i=1}^L$ a basis of $\chi_{U_L} M_0$ over $L^\infty(\mu \lfloor U_i)$; without loss of generality, we can assume that

$$(7.60) \quad \chi_{U_L} w = \sum_{j \in \Lambda_{k,N}} \lambda_j \tilde{m}_{j_1} \wedge \cdots \wedge \tilde{m}_{j_k},$$

with $\lambda_{(1, \dots, k)} \neq 0$. Moreover, by [Wea00, Thm. 10] we can choose a measurable $V \subset U_L$ with $\chi_V \lambda_{(1, \dots, k)} \neq 0$ and find $C > 0$ such that, if we define for $x \in V$

$$(7.61) \quad p_x : \mathbb{R}^L \rightarrow (0, \infty)$$

$$v \mapsto \left| \sum_{i=1}^L v_i \tilde{m}_i \right|_{M, \text{loc}}(x),$$

then p_x is a norm satisfying

$$(7.62) \quad C p_x(v) \geq \|v\|_\infty \quad (\forall (x, v) \in V \times \mathbb{R}^L).$$

Note that functions in $L^\infty(\mu \mathbb{L}V)$ can be canonically extended to $L^\infty(\mu)$ because we can indentify $L^\infty(\mu \mathbb{L}V)$ with $\chi_V L^\infty(\mu)$; the maps

$$(7.63) \quad \begin{aligned} \xi_i &: \chi_V M_0 \rightarrow L^\infty(\mu) \quad (i = 1, \dots, L) \\ \sum_{i=1}^L \lambda_i \tilde{m}_i &\mapsto \lambda_i, \end{aligned}$$

are bounded linear functionals by (7.62). By the Hanh-Banach Theorem [Wea00, Thm. 5] the $\{\xi_i\}$ can be extended to elements of M' ; in particular,

$$(7.64) \quad \begin{aligned} \Xi &: M^k \mapsto L^\infty(\mu) \\ (m_1, \dots, m_k) &\mapsto \det((\langle \xi_i, m_j \rangle)_{i,j=1}^k) \end{aligned}$$

defines an element of $\text{Alt}_k(M; L^\infty(\mu))$ and

$$(7.65) \quad E(w)(\chi_V \Xi) = \chi_V \lambda_{(1, \dots, k)} \neq 0$$

showing that E is injective. The proof is now completed as in Theorem 7.40. \square

We now provide a characterization of the norms in the exterior powers.

Lemma 7.66. *For Z a Banach space, if $w \in \bigwedge^k Z \hookrightarrow \text{Ext}^k Z$*

$$(7.67) \quad \|w\|_{\text{Ext}^k Z} = \inf \left\{ \sum_{i \in I} \|z_{i_1}\|_Z \cdots \|z_{i_k}\|_Z : w = \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k} \right\}.$$

If M is an $L^\infty(\mu)$ -normed module, for each

$$(7.68) \quad w \in L^\infty(\mu) \bigwedge^k M \hookrightarrow \text{Ext}_{\mu, \text{loc}}^k M,$$

$$(7.69) \quad \|w\|_{\text{Ext}_{\mu, \text{loc}}^k M} = \inf \left\{ \left\| \sum_{i \in I} |m_{i_1}|_{M, \text{loc}} \cdots |m_{i_k}|_{M, \text{loc}} \right\|_{L^\infty(\mu)} : \right. \\ \left. w = \sum_{i \in I} m_{i_1} \wedge \cdots \wedge m_{i_k} \right\};$$

moreover, if $w = \sum_{i \in I} m_{i_1} \wedge \cdots \wedge m_{i_k}$,

$$(7.70) \quad |w|_{\text{Ext}_{\mu, \text{loc}}^k M, \text{loc}} \leq \sum_{i \in I} |m_{i_1}|_{M, \text{loc}} \cdots |m_{i_k}|_{M, \text{loc}}.$$

Proof of Lemma 7.66. For Z a Banach space, define for $w \in \bigwedge^k Z$

$$(7.71) \quad \gamma(w) = \inf \left\{ \sum_{i \in I} \|z_{i_1}\|_Z \cdots \|z_{i_k}\|_Z : w = \sum_{i \in I} z_{i_1} \wedge \cdots \wedge z_{i_k} \right\};$$

then $\gamma(w)$ is a seminorm and (7.42) shows that

$$(7.72) \quad \|w\|_{\text{Ext}^k Z} \leq \gamma(w);$$

in particular (7.72) shows that γ is a norm on $\bigwedge^k Z$ and the same argument used in the proof of Theorem 7.40 (compare (7.48)) shows that the completion of $\bigwedge^k Z$ in the γ -norm satisfies the universal property characterizing $\text{Ext}^k Z$; thus $\|w\|_{\text{Ext}^k Z} = \gamma(w)$.

Let M an $L^\infty(\mu)$ -normed module; we first show (7.70). It suffices to show that for each U μ -measurable,

$$(7.73) \quad \|\chi_U w\|_{\text{Ext}_{\mu, \text{loc}}^k M} \leq \left\| \chi_U \sum_{i \in I} |m_{i_1}|_{M, \text{loc}} \cdots |m_{i_k}|_{M, \text{loc}} \right\|_{L^\infty(\mu)} ;$$

from the definition of $\|\cdot\|_{\text{Ext}_{\mu, \text{loc}}^k M}$ (proof of Theorem 7.54) we can find, for each $\varepsilon > 0$, an alternating map $\varphi \in \text{Alt}_k(M; L^\infty(\mu))$ with norm at most 1 and satisfying:

$$(7.74) \quad \|\chi_U w\|_{\text{Ext}_{\mu, \text{loc}}^k M} \leq \|\tilde{\varphi}(\chi_U w)\|_{L^\infty(\mu)} + \varepsilon;$$

but (7.30) implies

$$(7.75) \quad |\tilde{\varphi}(\chi_U w)| \leq \chi_U \sum_{i \in I} |m_{i_1}|_{M, \text{loc}} \cdots |m_{i_k}|_{M, \text{loc}},$$

from which we obtain (7.73) taking the essential sup and letting $\varepsilon \searrow 0$. To show (7.69) let

$$(7.76) \quad \gamma(w) = \inf \left\{ \left\| \sum_{i \in I} |m_{i_1}|_{M, \text{loc}} \cdots |m_{i_k}|_{M, \text{loc}} \right\|_{L^\infty(\mu)} : w = \sum_{i \in I} m_{i_1} \wedge \cdots \wedge m_{i_k} \right\};$$

then $\gamma(w)$ is a seminorm on $L^\infty(\mu) \bigwedge^k M$. Note that (7.69) implies $\|\cdot\|_{\text{Ext}_{\mu, \text{loc}}^k M} \leq \gamma$, so that γ is a norm; the proof of Theorem 7.54 implies that the completion Y of $L^\infty(\mu) \bigwedge^k M$ in the γ -norm satisfies the universal property defining $\text{Ext}_{\mu, \text{loc}}^k M$ provided that Y is an $L^\infty(\mu)$ -normed module. To show that Y is an $L^\infty(\mu)$ -normed module it suffices to show that for a μ -measurable set U ,

$$(7.77) \quad \gamma(w) = \max(\gamma(\chi_U w), \gamma(\chi_{U^c} w)).$$

Having shown (7.77), uniqueness of $\text{Ext}_{\mu, \text{loc}}^k M$ will imply that $\|\cdot\|_{\text{Ext}_{\mu, \text{loc}}^k M} = \gamma$. To show (7.77), for $\varepsilon > 0$ let

$$(7.78) \quad \chi_U w = \sum_{i \in I_U} \chi_U m_{i_1}^{(1)} \wedge \cdots \wedge \chi_U m_{i_k}^{(1)},$$

$$(7.79) \quad \chi_{U^c} w = \sum_{i \in I_{U^c}} \chi_{U^c} m_{i_1}^{(2)} \wedge \cdots \wedge \chi_{U^c} m_{i_k}^{(2)},$$

with

$$(7.80) \quad \left\| \sum_{i \in I_U} |\chi_U m_{i_1}^{(1)}|_{M, \text{loc}} \cdots |\chi_U m_{i_k}^{(1)}|_{M, \text{loc}} \right\|_{L^\infty(\mu)} < \gamma(\chi_U w) + \varepsilon$$

$$(7.81) \quad \left\| \sum_{i \in I_{U^c}} |\chi_{U^c} m_{i_1}^{(2)}|_{M, \text{loc}} \cdots |\chi_{U^c} m_{i_k}^{(2)}|_{M, \text{loc}} \right\|_{L^\infty(\mu)} < \gamma(\chi_{U^c} w) + \varepsilon;$$

without loss of generality (introducing null terms) we can assume that $I_U = I_{U^c} = I$ so that (7.77) follows observing that

$$(7.82) \quad w = \sum_{i \in I} (\chi_U m_{i_1}^{(1)} + \chi_{U^c} m_{i_1}^{(2)}) \wedge \cdots \wedge (\chi_U m_{i_k}^{(1)} + \chi_{U^c} m_{i_k}^{(2)})$$

and letting $\varepsilon \searrow 0$. □

There are also pairings between exterior powers:

Lemma 7.83. *Suppose Z is a Banach space; the bilinear mapping*

$$(7.84) \quad \wedge : \bigwedge^k Z \times \bigwedge^l Z \rightarrow \bigwedge^{k+l} Z$$

which on pairs of simple multivectors is given by:

$$(7.85) \quad \wedge : ((z_1, \dots, z_k), (u_1, \dots, u_l)) \mapsto z_1 \wedge \dots \wedge z_k \wedge u_1 \wedge \dots \wedge u_l,$$

extends to a bounded bilinear map

$$(7.86) \quad \wedge : \text{Ext}^k Z \times \text{Ext}^l Z \rightarrow \text{Ext}^{k+l} Z$$

satisfying

$$(7.87) \quad \|\omega_1 \wedge \omega_2\|_{\text{Ext}^{k+l} Z} \leq \|\omega_1\|_{\text{Ext}^k Z} \|\omega_2\|_{\text{Ext}^l Z}.$$

*Suppose M is an $L^\infty(\mu)$ -module; for $1 \leq i \leq k$, the bilinear mapping (in the category **Ban**)*

$$(7.88) \quad \begin{aligned} \Phi_i : L^\infty(\mu) \times \bigwedge^k M &\rightarrow \bigwedge^k M \\ \left(\lambda, \sum_{j \in J} m_{j_1} \wedge \dots \wedge m_{j_k} \right) &\mapsto \sum_{j \in J} m_{j_1} \wedge \dots \wedge \lambda m_{j_i} \wedge \dots \wedge m_{j_k} \end{aligned}$$

extends to a bounded bilinear map

$$(7.89) \quad \Phi_i : L^\infty(\mu) \times \text{Ext}^k M \rightarrow \text{Ext}^k M$$

satisfying

$$(7.90) \quad \|\Phi_i(\lambda, \omega)\|_{\text{Ext}^k Z} \leq \|\lambda\|_{L^\infty(\mu)} \|\omega\|_{\text{Ext}^k Z}.$$

Proof of Lemma 7.83. It follows from the first part of Lemma 7.66; in particular, (7.87) and (7.90) follow from (7.67). \square

We now turn to the existence of the exterior power in the category ${}^\infty_\mu \mathbf{Mod}$.

Theorem 7.91. *For M an $L^\infty(\mu)$ -module, the k -th exterior power in the category ${}^\infty_\mu \mathbf{Mod}$ exists and can be realized as a quotient space of $\text{Ext}^k M$ (in **Ban**) by the closure of the linear span of the set*

$$(7.92) \quad \left\{ \Phi_i(\lambda, \omega) - \Phi_j(\lambda, \omega) : 1 \leq i, j \leq k, \lambda \in L^\infty(\mu), \omega \in \bigwedge^k M \right\}.$$

Proof of Theorem 7.91. Let \mathcal{Q} denote the linear span of the set (7.92). If $\varphi \in \text{Alt}_k(M; N)$, where N is an $L^\infty(\mu)$ -module, let $\tilde{\varphi} : \text{Ext}^k M \rightarrow N$ denote the corresponding map given by the universal property of $\text{Ext}^k M$; note that $\tilde{\varphi}$ annihilates \mathcal{Q} . Moreover, $\text{Ext}^k M / \mathcal{Q}$ becomes an $L^\infty(\mu)$ -module letting

$$(7.93) \quad \lambda \cdot [\omega] = [\Phi_i(\lambda, \omega)] \quad ((\lambda, \omega) \in L^\infty(\mu) \times \text{Ext}^k M \text{ and } 1 \leq i \leq k).$$

If we let π' denote the composition of $\pi : M^k \rightarrow \text{Ext}^k M$ with the quotient map $\text{Ext}^k M \rightarrow \text{Ext}^k M / \mathcal{Q}$, then $\pi' \in \text{Alt}_k(M; \text{Ext}^k M / \mathcal{Q})$; similarly, if we let $\hat{\varphi} : \text{Ext}^k M / \mathcal{Q} \rightarrow N$ the map induced by $\tilde{\varphi}$, then $\hat{\varphi} \in \text{hom}(\text{Ext}^k M / \mathcal{Q}, N)$. Note that $\hat{\varphi} \circ \pi' = \varphi$ and that uniqueness of $\hat{\varphi}$ follows from uniqueness of $\tilde{\varphi}$. Finally, as $\|\omega\|_{\text{Ext}^k M / \mathcal{Q}} \leq \|\omega\|_{\text{Ext}^k M}$, $\|\hat{\varphi}\|_{\text{hom}(\text{Ext}^k M / \mathcal{Q}, N)} = \|\tilde{\varphi}\|_{\text{hom}(\text{Ext}^k M, N)} = \|\varphi\|_{\text{Alt}_k(M; N)}$. \square

Remark 7.94. Note that if M is an $L^\infty(\mu)$ -module, we have an \mathbb{R} -linear surjection

$$(7.95) \quad \text{Ext}^k M \rightarrow \text{Ext}_\mu^k M$$

with norm at most 1; similarly, if M is an $L^\infty(\mu)$ -normed module, we have an $L^\infty(\mu)$ -linear surjection

$$(7.96) \quad \text{Ext}_\mu^k M \rightarrow \text{Ext}_{\mu, \text{loc}}^k M$$

with norm at most 1.

7.2. Alberti representations in Banach spaces. In this Subsection we prove a refinement for the production of Alberti representations in Banach spaces when the speed and direction are specified using bounded linear maps.

Theorem 7.97. *Suppose that Z is a separable Banach space, μ is a Radon measure on Z and suppose that $f : Z \rightarrow \mathbb{R}^q$ and $g : Z \rightarrow \mathbb{R}$ are bounded linear maps. Let $\mathcal{C}(w, \alpha)$ be a q -dimensional cone field on Z and $\delta : Z \rightarrow (0, \infty)$ a Borel map; then the following are equivalent:*

- (1) *The measure μ admits an Alberti representation in the f -direction of $\mathcal{C}(w, \alpha)$ with g -speed $> \delta$.*
- (2) *The measure μ admits a $(\delta/\|g\|_{Z^*}, 1)$ -biLipschitz Alberti representation $\mathcal{A} = (P, \nu)$ in the f -direction of $\mathcal{C}(w, \alpha)$ with g -speed $> \delta$ and such that $\text{spt } P \subset \text{Curves}(Z)$ and $\nu_\gamma = h\Psi_\gamma$ where h is a Borel function on Z and*

$$(7.98) \quad \Psi_\gamma = \gamma_\# \mathcal{L}^1 \llcorner [0, 1].$$

Proof of Theorem 7.97. It suffices to show that (1) implies (2). For the moment, we assume that the functions w , α and δ are constant and that the set $\text{spt } \mu$ is compact. By rescaling g and δ , we can assume that $\|g\|_{Z^*} = 1$. Note that $\text{spt } \mu$ must contain a fragment γ with $(g \circ \gamma)'(t) > \delta \text{md } \gamma(t)$ and $(f \circ \gamma)'(t) \in \mathcal{C}(w, \alpha)$ for $\mathcal{L}^1 \llcorner \text{dom } \gamma$ -a.e. t . In particular, there is a vector $z \in Z$ in the unit sphere of Z satisfying $g(z) \geq \delta + 1/n_0$ and $f(z) \in \bar{\mathcal{C}}(w, \alpha - 1/n_0)$ for some n_0 . Let \mathcal{K} denote the closed convex hull of $\text{spt } \mu \cup (\text{spt } \mu + z)$ in Z and note that \mathcal{K} is compact. For $n \in \mathbb{N}$ let \mathcal{G}_n denote the compact set of all $(\delta, 1)$ -biLipschitz maps $\gamma : [0, 1] \rightarrow \mathcal{K}$ satisfying:

$$(7.99) \quad \text{sgn}(t - s) (f \circ \gamma(t) - f \circ \gamma(s)) \in \bar{\mathcal{C}}(w, \alpha - 1/n)$$

$$(7.100) \quad \text{sgn}(t - s) (g \circ \gamma(t) - g \circ \gamma(s)) \geq (\delta + 1/n)|t - s|.$$

Applying Lemma 2.56 in [Sch13] repeatedly, we obtain a decomposition $\mu = \mu' + \mu \llcorner F$ where μ' has an Alberti representation of the desired form and $F \subset \text{spt } \mu$ is an $F_{\sigma\delta}$ which is \mathcal{G}_n -null for every n .

We now show that for each fragment $\gamma \in \text{Frag}(\text{spt } \mu)$ in the f -direction of $\mathcal{C}(w, \alpha)$ and with g -speed $> \delta$, the set F is \mathcal{H}^1_γ -null; by (1), this will imply that $\mu(F) = 0$.

Let γ be such a fragment and assume that it is L -Lipshitz. Note that, if we find countably many compact sets $K_\alpha \subset \text{dom } \gamma$ with $\mathcal{H}^1_{\gamma|K_\alpha}(F) = 0$, then $\mathcal{H}^1_\gamma(F) = 0$. This allows to use Egorov and Lusin's Theorems to simplify the discussion. In particular, because \mathcal{K} is convex and because the functions f and g are linear, we can use the argument used to prove Theorem 2.64 in [Sch13] to reduce to the case in which, for some $\rho > 0$, the fragment γ extends to an $(L + \rho)$ -Lipschitz map $\tilde{\gamma} : I_\gamma \rightarrow Z$, where I_γ denotes the minimal interval containing $\text{dom } \gamma$, such that for some $n_1 \in \mathbb{N}$, the fragment $\tilde{\gamma}$ is in the f -direction of $\bar{\mathcal{C}}(w, \alpha - 1/n_1)$ with g -speed $\geq \delta + 1/n_1$. By precomposing $\tilde{\gamma}$ with an affine map and dividing I_γ into smaller

subintervals, we can reduce to the case in which $\tilde{\gamma}$ is 1-Lipschitz and $I_\gamma \subset [0, 1]$. Letting t_0 denote the right extremum of I_γ , we extend $\tilde{\gamma}$ to $[t_0, 1]$ by letting $\tilde{\gamma}|_{[t_0, 1]}$ be the segment joining $\tilde{\gamma}(t_0)$ to $(1 - t_0)(\tilde{\gamma}(t_0) + z)$. Note that $\text{md } \tilde{\gamma} \leq 1$ and, letting $n_2 = \max(n_0, n_1)$, we have $(g \circ \tilde{\gamma})' \geq \delta + 1/n_2$ and $(f \circ \tilde{\gamma}) \in \bar{\mathcal{C}}(w, \alpha - 1/n_2)$. In particular, $\tilde{\gamma} \in \mathcal{G}_{n_2}$ which implies $\mathcal{H}^1_{\tilde{\gamma}}(F) = 0$ and then $\mathcal{H}^1_\gamma(F) = 0$.

The case in which $\text{spt } \mu$ is not compact and the functions w , α and δ are not constant, is treated by using Egorov and Lusin's Theorems like in the last part of the proof of Theorem 2.64 in [Sch13]. \square

7.3. Renorming. The goal of this Subsection is the proof of the following result about renorming the module $\mathcal{X}(\mu)$ by taking a biLipschitz deformation of the metric on X .

Theorem 7.101. *Let (X, d) be a Polish space and μ a Radon measure on X . For each $\varepsilon > 0$ there is a metric $d^{(\varepsilon)}$ which satisfies*

$$(7.102) \quad d \leq d^{(\varepsilon)} \leq (1 + \varepsilon)d$$

and such that the corresponding local norm $|\cdot|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)}$ is strictly convex.

We now fix some notation that will be used through this Subsection. We let $\{\psi_n\}$ be a countable generating set for the Lipschitz algebra $\text{Lip}_b(X)$ and we assume that each function ψ_n is 1-Lipschitz. We then introduce the pseudometrics

$$(7.103) \quad \Psi(x, y) = \left\| \left(\frac{\psi_n(x) - \psi_n(y)}{n} \right)_n \right\|_{l^2}$$

$$(7.104) \quad \Psi_M(x, y) = \left(\sum_{n=1}^M \frac{(\psi_n(x) - \psi_n(y))^2}{n^2} \right)^{1/2},$$

and observe that $\Psi_M \leq \Psi \leq \frac{\pi}{\sqrt{6}}d$. We also define functions

$$(7.105) \quad \begin{aligned} \Phi : X &\rightarrow l^2 \\ x &\mapsto \left(\frac{\psi_n(x)}{n} \right)_n \end{aligned}$$

and

$$(7.106) \quad \begin{aligned} \Phi_M : X &\rightarrow \mathbb{R}^M \\ x &\mapsto \left(\frac{\psi_n(x)}{n} \right)_{n=1}^M, \end{aligned}$$

and observe that Φ and Φ_M are $\frac{\pi}{\sqrt{6}}$ -Lipschitz with respect to the distance d . We finally let

$$(7.107) \quad d^{(\varepsilon)} = d + \varepsilon \Psi$$

so that

$$(7.108) \quad d \leq d^{(\varepsilon)} \leq \left(1 + \varepsilon \frac{\pi}{\sqrt{6}} \right) d.$$

Note that, given a derivation D , after choosing a Borel representative for each $D\psi_n$, we obtain Borel maps⁹

$$(7.109) \quad \begin{aligned} D\Phi : X &\rightarrow l^2 \\ x &\mapsto \left(\frac{D\psi_n(x)}{n} \right)_n, \end{aligned}$$

and

$$(7.110) \quad \begin{aligned} D\Phi_M : X &\rightarrow \mathbb{R}^M \\ x &\mapsto \left(\frac{D\psi_n(x)}{n} \right)_{n=1}^M. \end{aligned}$$

We will now prove that the local norm $|\cdot|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)}$ corresponding to the distance $d^{(\varepsilon)}$ is strictly convex. We start with the following Lemma, which is essentially folklore and whose proof is included for completeness.

Lemma 7.111. *If $g \in C^1(\mathbb{R}^k)$ and the functions $\{\psi_i\}_{i=1}^k$ are in $\text{Lip}_b(X)$, then for any derivation $D \in \mathcal{X}(\mu)$ it follows that*

$$(7.112) \quad Dg(\psi_1, \dots, \psi_k) = \sum_{l=1}^k \frac{\partial g}{\partial y^l}(\psi_1, \dots, \psi_k) D\psi_l.$$

Proof of Lemma 7.111. The idea of the proof is essentially based on [AK00, Thm. 3.5(i)]. As the functions $\{\psi_i\}_{i=1}^k$ are bounded, letting $\psi : X \rightarrow \mathbb{R}^k$ the Lipschitz function whose i -th component is ψ_i , there is a k -dimensional simplex S^{10} centred about the origin such that $\psi(X)$ lies in the interior of S . Using that $g \in C^1(\mathbb{R}^k)$, it is possible to construct Lipschitz functions $g_n : S \rightarrow \mathbb{R}$ such that:

- (1) there is $M_n \in \mathbb{N}$ such that, if S^{M_n} denotes the M_n -th iterated barycentric subdivision of S , the function g_n is affine linear on each simplex $\Delta \in S^{M_n}$:

$$(7.113) \quad g_n(v) = \langle V_{n,\Delta}, v \rangle + c_{n,\Delta} \quad (v \in \Delta).$$

- (2) For each simplex $\Delta \in S^{M_n}$ one has

$$(7.114) \quad \sup_{v \in \Delta} |g(v) - g_n(v)| \leq \frac{1}{n}$$

$$(7.115) \quad \sup_{v \in \Delta} \|V_{n,\Delta} - \nabla g(v)\|_2 \leq \frac{1}{n}.$$

We now let

$$(7.116) \quad f(x) = g(\psi_1(x), \dots, \psi_k(x))$$

$$(7.117) \quad f_n(x) = g_n(\psi_1(x), \dots, \psi_k(x)),$$

and observe that as $f_n|_{\psi^{-1}(\Delta)}$ agrees with the function

$$(7.118) \quad x \mapsto \langle V_{n,\Delta}, \psi(x) \rangle + c_{n,\Delta},$$

the locality property of derivations implies that

$$(7.119) \quad Df_n(x) = \langle V_{n,\Delta}, D\psi(x) \rangle$$

for $\mu|_{\psi^{-1}(\Delta)}$ -a.e. x . As $f_n \xrightarrow{w^*} f$, (7.112) follows from (7.119) and (7.115). \square

⁹The Borel σ -algebras for the strong and the weak topologies on l^2 coincide

¹⁰we take simplices to be closed

The following Lemma is a key step in the proof of Theorem 7.101.

Lemma 7.120. *Let $F : X \rightarrow \mathbb{R}^M$ be Lipschitz, $D \in \mathcal{X}(\mu)$ and $\theta : X \rightarrow (0, \pi/2)$ a Borel map. Let*

$$(7.121) \quad V_F = \{x : DF(x) \neq 0\};$$

then $\mu \llcorner V_F$ admits an Alberti representation in the F -direction of $\mathcal{C}\left(\frac{DF}{\|DF\|_2}, \theta\right)$.

Proof of Lemma 7.120. The proof is essentially based on the argument used in Lemma 3.97 in [Sch13] and details are included for completeness. We consider a Borel $L^\infty(\mu \llcorner V_F)$ -partition of unity $\{V_l^{(0)}\}_{l \in \mathbb{N}}$ such that, for each l , there is a pair $(s_l, \theta_l) \subset (0, \infty) \times (0, \pi/2)$ with:

$$(7.122) \quad |D|_{\mathcal{X}(\mu \llcorner V_F), \text{loc}}(x) \in (s_l, 2s_l) \quad (\forall x \in V_l^{(0)})$$

$$(7.123) \quad \theta(x) \in (\theta_l, 2\theta_l) \quad (\forall x \in V_l^{(0)});$$

we further subdivide the $\{V_l^{(0)}\}_{l \in \mathbb{N}}$ to obtain a Borel $L^\infty(\mu \llcorner V_F)$ -partition of unity $\{V_l^{(1)}\}_{l \in \mathbb{N}}$ such that, for each l , (7.122) and (7.123) hold and there are $c_l > 0$ and $\varepsilon_l^{(1)} \in (0, c_l/2)$ such that:

$$(7.124) \quad \|DF(x)\|_2 \in (c_l, c_l + \varepsilon_l^{(1)}) \quad (\forall x \in V_l^{(1)});$$

note that the values of each $\varepsilon_l^{(1)}$ will be chosen later depending on the corresponding values of s_l and θ_l which were obtained in the previous step. We finally subdivide the $\{V_l^{(1)}\}_{l \in \mathbb{N}}$ to obtain a Borel $L^\infty(\mu \llcorner V_F)$ -partition of unity $\{V_l^{(2)}\}_{l \in \mathbb{N}}$ such that, for each l , (7.122), (7.123) and (7.124) hold and there are $w_l \in \mathbb{S}^{M-1}$ and $\varepsilon_l^{(2)} \in (0, \varepsilon_l^{(1)})$ such that:

$$(7.125) \quad \mathcal{C}(w_l, \theta_l/2) \subset \mathcal{C}\left(\frac{DF(x)}{\|DF(x)\|_2}, \theta_l\right) \quad (\forall x \in V_l^{(2)})$$

$$(7.126) \quad \left\| \frac{DF(x)}{\|DF(x)\|_2} - w_l \right\|_2 \leq \varepsilon_l^{(2)} \quad (\forall x \in V_l^{(2)});$$

note that the values of each $\varepsilon_l^{(2)}$ will be chosen later depending on the corresponding values of s_l , θ_l , c_l and $\varepsilon_l^{(1)}$ which were obtained in the previous steps. We now estimate the error in approximating DF by $c_l w_l$ on $V_l^{(2)}$:

$$(7.127) \quad \begin{aligned} \|DF - c_l w_l\|_2 &\leq \|DF - \|DF\|_2 w_l\|_2 + \| \|DF\|_2 w_l - c_l w_l \|_2 \\ &\leq \|DF\|_2 \left\| \frac{DF(x)}{\|DF(x)\|_2} - w_l \right\|_2 + \|DF\|_2 - c_l \\ &\leq \underbrace{(c_l + \varepsilon_l^{(1)})\varepsilon_l^{(2)} + \varepsilon_l^{(1)}}_{\eta_l}. \end{aligned}$$

In particular, if u is a unit vector orthogonal to w_l ,

$$(7.128) \quad \chi_{V_l^{(2)}} |D\langle u, F \rangle| = \chi_{V_l^{(2)}} |\langle u, DF - w_l c_l \rangle| \leq \frac{\eta_l}{s_l} |D|_{\mathcal{X}(\mu), \text{loc}}.$$

We now suppose that the Borel set $S_l \subset V_l^{(2)}$ is $\text{Frag}(X, F, \tilde{\delta}_l, w_l, \theta_l/2)$ -null; using (7.128) and Lemma 2.73 (compare also Lemma 3.68 and Lemma 3.76 in [Sch13] for details) we obtain

$$(7.129) \quad \chi_{S_l} |D\langle w_l, F \rangle| \leq \left(\tilde{\delta}_l + (M-1) \frac{\eta_l}{s_l} \cot(\theta_l/2) \right) |D|_{\mathcal{X}(\mu), \text{loc}};$$

on the other hand, we have

$$(7.130) \quad \chi_{V_l^{(2)}} D\langle w_l, F \rangle \geq \chi_{V_l^{(2)}} (c_l - \eta_l).$$

In particular, if $\mu(S_l) > 0$ we have

$$(7.131) \quad \tilde{\delta}_l \geq \frac{c_l - \eta_l}{2s_l} - (M-1) \frac{\eta_l}{s_l} \cot(\theta_l/2);$$

this implies that $\mu \llcorner V_l^{(2)}$ admits an Alberti representation \mathcal{A}_l in the F -direction of $\mathcal{C}(w_l, \theta_l/2)$ with F -speed

$$(7.132) \quad \geq \delta_l = \frac{c_l - 2\eta_l}{2s_l} - (M-1) \frac{\eta_l}{s_l} \cot(\theta_l/2),$$

provided that δ_l is positive. Note that

$$(7.133) \quad \delta_l = \frac{1}{2s_l} \left(c_l - 2(c_l + \varepsilon_l^{(1)})\varepsilon_l^{(2)} - 2\varepsilon_l^{(1)} \right) - (M-1) \frac{(c_l + \varepsilon_l^{(1)})\varepsilon_l^{(2)} + \varepsilon_l^{(1)}}{s_l} \cot(\theta_l/2);$$

if at each step the $\varepsilon_l^{(1)}$ and $\varepsilon_l^{(2)}$ are chosen sufficiently small, one can ensure that $\delta_l > 0$. The proof is completed by gluing together the $\{\mathcal{A}_l\}$ (Theorem 2.32) and using (7.125). \square

Lemma 7.134. *The local norms $|\cdot|_{\mathcal{X}(\mu), \text{loc}}$ and $|\cdot|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)}$ are related by the following equation:*

$$(7.135) \quad |D|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)} = |D|_{\mathcal{X}(\mu), \text{loc}} + \varepsilon \|D\Phi\|_{l^2} \quad (\forall D \in \mathcal{X}(\mu)).$$

Proof of Lemma 7.134. We first show that

$$(7.136) \quad |D|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)} \leq |D|_{\mathcal{X}(\mu), \text{loc}} + \varepsilon \|D\Phi\|_{l^2}$$

by showing that, for each $x \in X$, the distance function $d^{(\varepsilon)}(x, \cdot)$ satisfies

$$(7.137) \quad \left| Dd^{(\varepsilon)}(x, \cdot) \right| \leq |D|_{\mathcal{X}(\mu), \text{loc}} + \varepsilon \|D\Phi\|_{l^2}.$$

Without loss of generality, we can assume that X is bounded. Let $d_M^{(\varepsilon)} = d + \varepsilon \Psi_M$ and observe that the sequence of Lipschitz functions $\{d_M^{(\varepsilon)}(x, \cdot)\}_{M \in \mathbb{N}}$ converges to $d^{(\varepsilon)}(x, \cdot)$, in the weak*-topology, as $M \nearrow \infty$. As $d(x, \cdot)$ is 1-Lipschitz with respect to d , we have:

$$(7.138) \quad |Dd(x, \cdot)| \leq |D|_{\mathcal{X}(\mu), \text{loc}}.$$

On the closed set $C_0 = \{y : \Psi_M(x, y) = 0\}$, one has $D\Psi_M(x, \cdot) = 0$ by locality of derivations. For $\delta > 0$ consider the closed set $C_\delta = \{y : \Psi_M(x, y) \geq \delta\}$. We can find a function $g : \mathbb{R}^M \rightarrow (0, \infty)$ of class $C^1(\mathbb{R}^M)$ such that, if for a $v \in \mathbb{R}^M$ one has

$$(7.139) \quad \left(\sum_{n=1}^M \frac{|v_n|^2}{n^2} \right)^{1/2} \geq \frac{\delta}{2},$$

then

$$(7.140) \quad g(v) = \left(\sum_{n=1}^M \frac{|v_n|^2}{n^2} \right)^{1/2}.$$

In particular, on C_δ , the function $\Psi_M(x, \cdot)$ coincides with

$$(7.141) \quad g(\psi_1(\cdot) - \psi_1(x), \dots, \psi_M(\cdot) - \psi_M(x)),$$

and Lemma 7.111 gives

$$(7.142) \quad D\Psi_M(x, y) = \frac{1}{\Psi_M(x, y)} \sum_{n=1}^M \frac{\psi_n(y) - \psi_n(x)}{n} \frac{D\psi_n(y)}{n}$$

for $\mu \ll C_\delta$ -a.e. y . Using the Cauchy inequality and a sequence $\delta_n \searrow 0$, we conclude that

$$(7.143) \quad |D\Psi_M(x, \cdot)| \leq \|D\Phi_M\|_2.$$

Combining (7.138) and (7.143) we obtain (7.137) and so (7.136) is proved.

We now show that

$$(7.144) \quad |D|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)} \geq |D|_{\mathcal{X}(\mu), \text{loc}} + \varepsilon \|D\Phi\|_{l^2},$$

and we will assume that a Borel representative has been chosen for each $D\psi_n$. We first consider the Borel set V_0 where $\|D\Phi\|_{l^2} = 0$. Having fixed $\eta > 0$, we take a Borel $L^\infty(\mu \ll V_0)$ -partition of unity $\{U_\alpha\}$ such that, for each α , there is a function f_α which is 1-Lipschitz with respect to the distance d and satisfying:

$$(7.145) \quad \chi_{U_\alpha} Df_\alpha \geq (1 - \eta) \chi_{U_\alpha} |D|_{\mathcal{X}(\mu), \text{loc}};$$

this implies that

$$(7.146) \quad \chi_{V_0} |D|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)} \geq (1 - \eta) \chi_{V_0} |D|_{\mathcal{X}(\mu), \text{loc}}.$$

We now consider the Borel set V_1 where $\|D\Phi\|_{l^2} > 0$. For each $\eta > 0$, we take an $L^\infty(\mu \ll V_1)$ -partition of unity $\{U_\alpha\}$, where each set U_α is compact and such that for each α there is a quadruple $(f_\alpha, M_\alpha, \theta_\alpha, \delta_\alpha)$ satisfying:

(P1): The function f_α is 1-Lipschitz with respect to the distance d , M_α is a natural number, $\theta_\alpha \in (0, \pi/2)$, and $\delta_\alpha > 0$.

(P2): The following inequality holds

$$(7.147) \quad \chi_{U_\alpha} Df_\alpha \geq (1 - \eta) \chi_{U_\alpha} |D|_{\mathcal{X}(\mu), \text{loc}}.$$

(P3): The Borel functions $\|D\Phi\|_{l^2}$ and $\|D\Phi_{M_\alpha}\|_2$ are continuous on U_α and satisfy

$$(7.148) \quad \|D\Phi_{M_\alpha}\|_2 \geq (1 - \eta) \|D\Phi\|_{l^2} \geq \delta_\alpha > 0.$$

(P4): For all $x, y \in U_\alpha$, if $u \in \mathcal{C} \left(\frac{D\Phi_{M_\alpha}(x)}{\|D\Phi_{M_\alpha}(x)\|_2}, 2\theta_\alpha \right) \cap \mathbb{S}^{M_\alpha-1}$, then

$$(7.149) \quad \langle u, D\Phi_{M_\alpha}(y) \rangle \geq (1 - \eta) \|D\Phi_{M_\alpha}(y)\|_2.$$

By Lemma 7.120 the measure $\mu \ll U_\alpha$ admits an Alberti representation in the f_α -direction of the cone field $\mathcal{C} \left(\frac{D\Phi_{M_\alpha}(x)}{\|D\Phi_{M_\alpha}(x)\|_2}, \theta_\alpha \right)$; in particular, for $\mu \ll U_\alpha$ -a.e. x , there is a fragment $\gamma_x \in \text{Frag}(U_\alpha)$ such that:

- (1) 0 is a Lebesgue density point of $\text{dom } \gamma_x$ and $\gamma_x(0) = x$.

(2) There is a $v_x \in \mathcal{C} \left(\frac{D\Phi_{M_\alpha}(x)}{\|D\Phi_{M_\alpha}(x)\|_2}, \theta_\alpha \right)$ with

$$(7.150) \quad \Phi_{M_\alpha}(\gamma(r)) = \Phi_{M_\alpha}(x) + v_x r + o(r).$$

In particular, there are $r_x, R_x > 0$ such that for each $y \in B(x, R_x) \cap U_\alpha^{11}$, one has

$$(7.151) \quad \frac{\Phi_{M_\alpha}(\gamma_x(r_x)) - \Phi_{M_\alpha}(y)}{\|\Phi_{M_\alpha}(\gamma_x(r_x)) - \Phi_{M_\alpha}(y)\|_2} \in \mathcal{C} \left(\frac{D\Phi_{M_\alpha}(x)}{\|D\Phi_{M_\alpha}(x)\|_2}, 2\theta_\alpha \right).$$

Let

$$(7.152) \quad \tilde{f}_\alpha = f_\alpha - \varepsilon \Psi_{M_\alpha}(\gamma_x(r_x), \cdot),$$

and observe that \tilde{f}_α is 1-Lipschitz with respect to the distance $d^{(\varepsilon)}$ and that

$$(7.153) \quad D\tilde{f}_\alpha = Df_\alpha - \varepsilon D\Psi_{M_\alpha}(\gamma_x(r), \cdot);$$

an argument similar to that used to prove (7.142) shows that for $\mu \llcorner (U_\alpha \cap B(x, R_x))$ -a.e. y ,

$$(7.154) \quad D\Psi_{M_\alpha}(\gamma_x(r_x), y) = - \frac{\langle \Phi_{M_\alpha}(\gamma_x(r_x)) - \Phi_{M_\alpha}(y), D\Phi_{M_\alpha}(y) \rangle}{\|\Phi_{M_\alpha}(\gamma_x(r_x)) - \Phi_{M_\alpha}(y)\|_2} \leq -(1 - \eta) \|D\Phi_{M_\alpha}\|_2,$$

where in the last step we used (7.151) and **(P4)**. Combining (7.154) with **(P2)** we obtain

$$(7.155) \quad \chi_{U_\alpha} D\tilde{f}_\alpha \geq (1 - \eta) \chi_{U_\alpha} |D|_{\mathcal{X}(\mu), \text{loc}} + \varepsilon (1 - \eta)^2 \chi_{U_\alpha} \|D\Phi\|_{l^2},$$

which implies

$$(7.156) \quad \chi_{V_1} |D|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)} \geq (1 - \eta) \chi_{V_1} |D|_{\mathcal{X}(\mu), \text{loc}} + \varepsilon (1 - \eta)^2 \chi_{V_1} \|D\Phi\|_{l^2};$$

letting $\eta \searrow 0$ in (7.156) and (7.147), (7.144) follows. \square

Proof of Theorem 7.101. Because of (7.108), we just need to show that the local norm $|\cdot|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)}$ associated to $d^{(\varepsilon)}$ is strictly convex. Consider derivations $D_1, D_2 \in \mathcal{X}(\mu)$ and suppose that for $\mu \llcorner U$ a.e. $x \in U$ one has:

$$(7.157) \quad |D_1 + D_2|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)}(x) = |D_1|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)}(x) + |D_2|_{\mathcal{X}(\mu), \text{loc}}^{(\varepsilon)}(x);$$

by Lemma 7.134 we have

$$(7.158) \quad \begin{aligned} |D_1 + D_2|_{\mathcal{X}(\mu), \text{loc}}(x) + \varepsilon \|D_1\Phi(x) + D_2\Phi(x)\|_{l^2} &= |D_1|_{\mathcal{X}(\mu), \text{loc}}(x) + \varepsilon \|D_1\Phi(x)\|_{l^2} \\ &\quad + |D_2|_{\mathcal{X}(\mu), \text{loc}}(x) + \varepsilon \|D_2\Phi(x)\|_{l^2}; \end{aligned}$$

because

$$(7.159) \quad |D_1 + D_2|_{\mathcal{X}(\mu), \text{loc}} \leq |D_1|_{\mathcal{X}(\mu), \text{loc}} + |D_2|_{\mathcal{X}(\mu), \text{loc}}$$

$$(7.160) \quad \|D_1\Phi + D_2\Phi\|_{l^2} \leq \|D_1\Phi\|_{l^2} + \|D_2\Phi\|_{l^2},$$

after choosing Borel representatives of $D_1\Phi$ and $D_2\Phi$, we find a Borel $V \subset U$ with $\mu(U \setminus V) = 0$ and such that:

$$(7.161) \quad \|D_1\Phi(x) + D_2\Phi(x)\|_{l^2} = \|D_1\Phi(x)\|_{l^2} + \|D_2\Phi(x)\|_{l^2} \quad (\forall x \in V).$$

¹¹the ball can be taken either with respect to d or $d^{(\varepsilon)}$.

The strict convexity of the norm on l^2 implies that for each $x \in V$ the vectors $D_1\Phi(x)$ and $D_2\Phi(x)$ are linearly dependent. Let

$$(7.162) \quad \tilde{V}_1 = \{(x, \lambda) \in V \times [-1, 1] : D_1\Phi(x) = \lambda D_2\Phi(x)\}$$

$$(7.163) \quad \tilde{V}_2 = \{(x, \lambda) \in V \times [-1, 1] : D_2\Phi(x) = \lambda D_1\Phi(x)\};$$

then \tilde{V}_1 and \tilde{V}_2 are Borel subsets of $X \times [-1, 1]$ and, denoting by V_i the projection of \tilde{V}_i on X , we have $V = V_1 \cup V_2$. Note that for each x the section $(\tilde{V}_i)_x$ is compact; in particular, by the Lusin-Novikov Uniformization Theorem [Kec95, Thm. 18.10] 18.10, the sets V_1 and V_2 are Borel and admit Borel uniformizing functions $\sigma_i : V_i \rightarrow [-1, 1]$. In particular,

$$(7.164) \quad \chi_{V_1} D_1\Phi = \sigma_1 \chi_{V_1} D_2\Phi$$

$$(7.165) \quad \chi_{V_2} D_2\Phi = \sigma_2 \chi_{V_2} D_1\Phi;$$

as the $\{\psi_n\}$ generate $\text{Lip}_b(X)$, (7.164) and (7.165) imply that $\chi_U D_1$ and $\chi_U D_2$, regarded as elements of $\mathcal{X}(\mu\text{LU})$, are linearly dependent. \square

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